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المملكة العربية السعودية
وزارة التعليم العالي
جامعة أم القرى
كلية العلـــوم التطبيقية
قسم العلـوم الرياضـية

# الآثار المعدلة ومجموع الدوال الذاتيـــة للمؤثرات التفاضلية من الرتبة الرابعـــة ذات الشروط الحدية الدورية وغير الدورية

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بحث تكميلي
لمتطلبات الحصول على درجة الماجستير
في الرياضيات البحتة ( تحليل دالي )

١٤٢٥هـ/ ٢٠٠٤م

وزارة التعليم العالي
جامعة أم القرى
كلية العلوم التطبيقية

## بسم الله الرحمن الرحيم

## محضر مناقشة رسالة ( ماجستير )

الحمد لله والصلاة والسلام على رسول الله صلى الله عليه وسلم وعلى اله وصحبه ومن والاه ففي تمام الساعة العاشرة من يوم الاربعاء..... الموافق ٢٤/٤/١٤٢٨ اجتمعت اللجنة المشكلة بقرار مجلس كلية رقم .....٦..... وفي جلسته ..٨.. بتاريخ ٢٦/٢/١٤٢٨هـ والمكونة من أصحاب السعادة :

| م | الاسم | الجهة التابع لها | وضع المناقش في اللجنة |
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| ١- | د. صالح عبد العزيز محمد صالح | جامعة أم القرى | مقرر الجلسة ومشرف |
| ٢- | د. مصطفى أحمد بيدري | جامعة أم القرى | مناقش داخلي |
| ٣- | د. حسن أحمد حسن زيدان | كلية المعلمين بالدمام | مناقش خارجي |

لمناقشة الرسالة المقدمة لنيل درجة الماجستير في العلوم الرياضية ..
من الطالب / الطالبة ......سارة محمد عبد الغفار سيحي......
تخصص ......تحليل دالي...... بكلية العلوم التطبيقية بمكة المكرمة بجامعة أم القرى .
بعنوان ...الارتباطات الطبيعية وجميع الدوال الناعمة للمتباينات التفاضلية من الرتبة الرابعة تحت الشروط الحدية الدورية وغير الدورية.
وبعد الانتهاء اللجنة من المناقشة في الساعة .....١٢.. ظهراً والتداول فيما بينها .
وبناء على موقف الطالب / الطالبة أثناء المناقشة
أوصت اللجنة بمنح درجة الماجستير في العلوم الرياضية بتقدير جيد جداً ١...... بالدرجة ..٨٨,٦٠٪.. الموضحة . والله الموفق .
ما اكتسبه من تقديرات بالسنة المنهجية ......٢٤,٢٤٪ + ٦٤,٤٪ من تقدير الرسالة .
ما اكتسبه من الساعات المعتمدة المطلوبة لهذه الدرجة ......٢٤ ساعة معتمدة .

**ملاحظات**

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| م | الاسم | الدرجة | التوقيع |
|---|---|---|---|
| ١- | د. صالح عبد العزيز محمد صالح | ٩١ | |
| ٢- | د. مصطفى أحمد بيدري | ٩٢ | |
| ٣- | د. حسن أحمد حسن زيدان | ٩٣ | |
| | مجموع الدرجات | ٢٧٦ | |
| | المتوسط | ٩٢ | |

رئيس قسم العلوم الرياضية

د . محمد بن عبدالغني دويخ

شرف...

جامعة أم القرى - كلية العلوم التطبيقية
قسم العلوم الرياضية
صورة طبق الأصل

## تقرير عن رسالة ( الماجستير ) بعد المناقشة

الحمد لله والصلاة والسلام على رسول الله وعلى آله وصحبه ومن والاه... وبعد:
في الساعة ( العاشرة ) من ( ☑ صباح ☐ مساء) يوم ( الأربعاء ) وتاريخ ٢٤ / ٢ / ١٤٢٥هـ الموافق ١٤ / ٤ / ٢٠٠٤م اجتمعت اللجنة المشكلة بقرار مجلس عمادة الدراسات العليا في جلسته رقم ( ١٢ ) وتاريخ ٢٥ / ١ / ١٤٢٥هـ المكونة من:

| م | الاسم | الرتبة العلمية | نوع المشاركة | ملحوظات |
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| ١ | د. صالح عبد العزيز محمد صالح | أستاذ مشارك | مقررا | |
| ٢ | د. مسعود أحمد بيري | أستاذ مساعد | عضوا | |
| ٣ | د. سعد أحمد سعد زيدان | أستاذ مشارك | عضوا | |
| ٤ | | | | |
| ٥ | | | | |

لمناقشة الرسالة العلمية المقدمة من الطالب / سارة محمد عبد الغفار رسلي.
ضمن متطلبات برنامج ( الماجستير ) في ( الرياضيات البحتة ) تخصص ( تحليل دالي ) وعنوانها
الأدوار المعممة ومجموعة الحلول الدائمة للتراجعات التفاضلية من الرتبة الأولى ذات الشروط الحدية الدورية وغير الدورية

وبعد انتهاء اللجنة من المناقشة في الساعة ( الثانية عشرة ظهرا ) والتداول فيما بينها وبناء على موقف الطالب في المناقشة حسب التقرير المرفق أوصت اللجنة بما يلي:
☑ قبول الرسالة بدرجة ( ٩٢ %) وتقدير ( ممتاز ) والتوصية بمنح الطالب درجة ( الماجستير ) في تخصص ( تحليل دالي )
☐ قبول الرسالة بدرجة ( / %) وتقدير ( / ) مع إجراء بعض التعديلات عليها ويفوض عضو المناقشة الدكتور/ بالتوصية بمنح الطالب درجة ( ) تخصص ( / ) بعد التأكد من الأخذ بالتعديلات المشار إليها في مدة لا تتجاوز ثلاثة أشهر من إعداد هذا التقرير.
☐ استكمال أوجه النقص في الرسالة، وإعادة مناقشتها خلال الفترة التي سيتم تحديدها من قبل مجلس عمادة الدراسات العليا، بناء على توصية مجلس القسم المختص.
☐ عدم قبول الرسالة.
☐ توصيات أخرى: ......................................
وبناء عليه حرر هذا التقرير في الساعة ( الواحدة ) من يوم ( الأربعاء ) الموافق ٢٤ / ٢ / ١٤٢٥هـ
نسأل الله تعالى أن يوفقه لما فيه خير الإسلام والمسلمين.

أسماء وتوقيعات اللجنة:

| الاسم | الدرجة من ( مائة ) | التوقيع |
|---|---|---|
| د. صالح عبد العزيز محمد صالح | ٩١ | |
| د. مسعود أحمد بيري | ٩٢ | |
| د. سعد أحمد سعد زيدان | ٩٢ | |
| | | |
| | | |

المتوسط الحسابي لمجموع الدرجات: ( ٩٢ )

اثنان وتسعون درجة

جامعة أم القرى - كلية العلوم التطبيقية - قسم العلوم الرياضية

بسم الله الرحمن الرحيم

# ملخـــص الرســـــــالة

## الآثار المعدلة ومجموع الدوال الذاتية للمؤثرات التفاضلية
## من الرتبة الرابعة ذات الشروط الحدية الدورية وغير الدورية

من المعروف أن مجموع عناصر قطر المصفوفة المربعة يساوي مجموع القيم الذاتية للمؤثر الخطي المعرف في الفراغ محدود الأبعاد أو بعبارة أخرى أثر المصفوفة يساوي الأثر الطيفي في الفراغ النوني وهذه النظرية محققة في المؤثرات النووية المعرفة في فراغ هلبرت .

ومن الطبيعي أن ينبثق سؤال حول تحقيق هذه النظرية في حالة المؤثرات غير المحددة وخاصة في حالة المؤثرات التفاضلية يكون أثر المصفوفة ،الأثر الطيفي غير موجودًا ، لهذا السبب عُرف مايسمى بالأثر المعدل .

ومن الجدير بالذكر إن دراسة الأثر المعدل للمؤثرات التفاضلية يلعب دورًا هامًا في مجالات عديدة مثل ( التحليل الرياضي ، الفيزياء النظرية ، وميكانيكا الكم ) كمايمكن إستخدام الأثر المعدل في المسائل العكسية في التحليل الدالي .

لقد قام عدد كبير من الباحثين بإيجاد الصيغ المختلفة للأثر المعدل للمؤثرات التفاضلية أمثال أ . م . جلفاند ، ب . م . ليفتيان [1] ، م . أ . نايمارك [2] ، جـ . أ . شارلز ، جـ . ر . هالبرج ، ف . أ . كرامر [3] ، ف . ي . ليدسكي ، ف . أ صدوفنتشي [4] ، [5] ، ف . أ صدوفنتشي ، ف . أ . ليوبشكين [6] ، ي . بلعباسي [7] ، ص . أ . صالح [8] ، أ . أ . درويش [9] ، أ . أ . درويش ، ص . أ . صالح [10] ، ص . أ . صالح ، ح . أ . زيدان [11] ، ص . أ . صالح ، م . أ . قاسم [12] ، أ . أ . درويش ، أ . أ . عبدالعال [13] ، ص . أ . صالح ، ر . م . علام [14] ، أ . س . بشنتسوف [15] ، د . ميلينكوفيك [16] ، ف . أ . ليوبشكين [17] ، ف . أ . صدوفنتشي ، ف . ف . دوبروفسكي [19] ، ص . أ . صالح [28] وآخرين .

إن الهدف الرئيسي لهذه الرسالة هو حساب الأثر المعدل ، مجموع الدوال الذاتية للمؤثرات التفاضلية من الرتبة الرابعة تحت الشروط الحدية الدورية وغير الدورية .

**تنقسم الرسالة الى مقدمة وثلاث أبواب :**

**الباب الأول :** يقدم بعض التعريفات والنظريات الأساسية التي تعتبر أساسًا رياضيًا للبابين الثاني والثالث وطبقاً لذلك قمنا بكتابة ملخص للآتي :

١ - الدوال التحليلية ، النقاط الشاذة ، البواقي ، نظرية كوشي ، نظرية روشَ .

٢ - الشروط الحدية ، بناء دالة جرين للمؤثرات التفاضلية من الرتبة النونية ثم إيجاذ دالة جرين للمؤثر $L - \lambda I$

٣ - الخواص الأساسية لدوال زيتا - ريمان $\zeta(s), \zeta(s,a)$

**الباب الثاني :** يختص بالخواص الأساسية للحلول والقيم الذاتية والدوال الذاتية للمسألة الأولى عندما تكون الشروط الحدية غير دورية ولذلك قمنا بالآتي :

١ - تقسيم المستوى الى قطاعات ، وتعريف القطاعين $S, T$

٢ - إيجاد الصيغ التقاربية للحلول الاساسية ومشتقاتها لمعادلة تفاضلية من الرتبة الرابعة .

٣ - إيجاد الصيغ التقاربية للقيم الذاتية ، لمعادلة تفاضلية من الرتبة الرابعة تحت شروط حدية غير دورية .

٤ - الحصول على الأثر المعدل لمؤثر تفاضلي من الرتبة الرابعة عندما تكون الشروط الحدية غير دورية .

٥ - إيجاد مجموع الدوال الذاتية لمؤثر تفاضلي من الرتبة الرابعة عندما تكون الشروط الحدية غير دورية .

**الباب الثالث :** قمنا بدراسة تفصيلية لمؤثر تفاضلي من الرتبة الرابعة تحت شروط حدية دورية وقد حصلنا على النتائج الآتية :

١ - الصيغ التقاربية للقيم الذاتية .

٢ - الأثر المعدل للقيم الذاتية ( بطريقتين )

٣ - مجموع الدوال الذاتية .

نأمل أن تكون هذه الرسالة مرجعاً للمهتمين بدراسة الأثر المعدل ومجموع الدوال الذاتية لمؤثرات تفاضلية من الرتبة الرابعة تحت شروط دورية وغير دورية ونحاول في الدراسات القادمة أن نعالج دراسة الأثر المعدل ومجموع القيم الذاتية لمؤثرات تفاضلية من الرتبة الرابعة تحت شروط عامة وكذلك دراسة الأثر المعدل ومجموع القيم الذاتية لمؤثرات تفاضلية من الرتبة النونية .

# SUMMARY

## Regularized Traces and Sum of Eigenfunctions of Differential Operators with Fourth Order Under the Periodic and Non-Periodic Conditions


The main role of this thesis is calculating the regularized traces of eigenvalues and sum of eigenfunctions of differential operator of fourth order , when the boundary conditions are periodic and the boundary conditions are non-periodic.

The thesis consists of three chapters :

*In chapter I :* We introduce some fundamental definitions and theorems which considered the necessary background mater for the other two chapters .

*In chapter II :* We constructed the domains S and T , the asymptotic formulae for fundamental solutions of the differential equation in the form

$$y^{(4)} + q(x)y = \lambda y, x \in [0, \pi]$$

are obtained.

Also , the asymptotic behavior of the eigenvalues for the given problem are studied , we obtain the regularized traces of the given problem by two methods . At the last of this chapter , we obtained the regularized sum of eigenfunctions of the given problem .

*In chapter III :* We studied the regularized traces and sum of eigenfunctions for periodic boundary conditions .

Kingdom of Saudi Arabia

**Ministry of Higher Education**

Umm Al-Qura University

Faculty of Applied Sciences

Department of Mathematical Sciences

# Regularized Traces and Sum of Eigenfunctions of Differential Operators with Fourth Order Under the Periodic and Non-Periodic Conditions

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A Thesis Submitted for the Degree of Master of Pure Mathematics (Functional Analysis)

**1425 / 2004**

بسم الله الرحمن الرحيم

اعزم وكد فإن مضيت فلا تقف

واصبر وثابر فالنجاح محقق

ليس الموفق من تواتيه المنى

لكن من رزق الثبات موفق

خليل مطران

# شكر وتقدير

قال تعالى: ﴿ لئن شكرتم لأزيدنكم ﴾ (سورة إبراهيم: ٧)

الحمد لله حمداً كثيراً طيباً مباركاً فيه ، غير مكفيّ ولا مستغنى عنه ، على ما أنعمه علي من إتمامي لهذا البحث .

ويسعدني أن أتقدم بجزيل الشكر والعرفان لسعادة الأستاذ الدكتور : صالح عبد العزيز صالح على كل ما قام به من توجيه ودعم وإرشاد خلال إعداد هذا البحث .

كما أتقدم بالشكر إلى كلية العلوم الرياضية بجامعة أم القرى لإتاحتها الفرصة لاستكمالي دراستي والحصول على درجة الماجستير في الرياضيات البحتة .

كما أشكر جميع أساتذتي اللذين كان لهم الفضل بعد الله عز وجل في إتمام هذه الدراسة ، والله أسأل أن يجزي الجميع خير الجزاء إنه ولي ذلك والقادر عليه .

آميـــن

# الإهداء

- **أُهدي هذا البحث إلى أبي....**

فقلب الأب هو هبة الله الرائعة

- **أُهدي هذا البحث إلى أمي ....**

فحب الأم لا يشيخ أبداً

- **أُهدي هذا البحث إلى عمي عبد الرحمن ....**

فكل سعي سيجزي الله ساعيه ، هيهات يذهب

سعي المحسنين هباء

# CONTENTS

# INTRODUCTION

It is well known that the summation of the diagonal elements in a square matrix is equal to the summation of eigenvalues of linear operator in finite dimensional space . In other words, the trace of a matrix is equal to the spectral trace in n-dimensionl space.

It is worth mentioning that this theoem is satisfied also in the case of nuclear operators, which are defined in Hilbert space [ 30 ] . Thus we might ask the following question : Is this theorem applicable in case of unbounded operators ?; especially in the case of differential operators the trace of matrices and spectral trace are not exist, for this reason we define the so-called "Regularized trace" .The study of regular trace for differential operators plays an important role in several fields such as mathematical analysis , theoretical physics and quantum mechanics. We can use also the regular trace in the inverse spectral problems in functional analysis .

Agood number of works has been devoted to the deduction of the formulae of regularized traces of differential operators such as I.M.Gelfand , B.M.Levitan (1953) [ 1 ], M.A.Naimark (1954) [ 2 ], J.A.Charles, J R . Halberg and

V.A.Kramer (1960) [ 3 ], V.B. Lidsky, V. A . Sadovnichii (1967, 1968) [4], [5], V.A.Sadovnichii , V.A.Lyubishkin. (1981) [6], Y.Belabbaci (1981) [7], S.A.Saleh (1984) [8], S.A.Saleh , H.A.Zedan (1987) [11], S.A.Saleh, M.A.Kassem (1987) [12], A . A. Darwish (1984) [9], A . A . Darwish , S. A. Saleh (1986) [10], A . A. Darwish , E. E . Abd-Aal. (1987) [13], S . A . Saleh, R.M. Allam ( 1991 ) [14], D.Milinkovic. (1991) [16], V.A.Lyubishkin (1991, 1993) , [19], [20], A.S. Pechentsov. ( 1991 , 1992 ), [15], [18], V.A.Sadovnichii, V.V.Dubrovskii (1992, 1996) [19], [21], [15], S.A.Saleh (2000) [28], and others .

The concept of regularized sum for eigenfunction of differential operators are introduced by S.A.Saleh (1998) , [22]. Also the proof of the expansion theorem of eigenfunction for multi-point and integral condition is given by S.A.Saleh,M.A.Kassem.(1998) [31].

The main role of this thesis is calculating the regularized traces of eigenvalues and sum of eigenfunctions of differential operator of fourth order , when the boundary conditions are periodic and the boundary conditins are non-periodic i-e, the differential equation

$y^{(4)} + q(x)\, y = \lambda\, y \; ; x \in [0,\pi]$

under the non-periodic conditions

$y(0) = y''(0) = y(\pi) = y''(\pi) = 0$ .

And also under the periodic conditions namely

$y(0) - y(\pi) = y'(0) - y'(\pi) = y''(0) - y''(\pi) = y'''(0) - y'''(0) = 0$

The thesis consists of three chapters , in the first chapter,we introduce some fundamental definitions and theorems which considered the necessary background mater for the other two chapters. Accordingly , we wirte short notes on the following :

1. Analytic functions , singular points , the residues , the poles , the Cauchy - Goursat theorem and Rouche$'s$ thearem
2. Linear differential expression .
3. The problem of inverting a differential operater.
4. Construction of Gree$n's$ function .
5. Rieman$n's$ Zeta functions $\zeta$ (s) , $\zeta$ (s,a) .

In chapter II , we constructed the domains Sand T,the asymptotic formulae for fundamental solutions of the differential equation in the form

$$y^{(4)}(x) + q(x)\,y = \lambda\, y \ , \qquad x \in [\,0 ,\pi\,]$$

are obtained .

Also , the asymptotic behaviour of the eigenvalues for the given problem are studied , ( theorem 2. 2 ) We obtain the

regularized traces of the given problem by two methods (theorem 2.3 ). At the last of chapter II , we obtained the regularized sum of eigenfunction of the given problem (theorem 2.4).

In chapterIII , we studied the regularized traces and sums of eigenfunctions for periodic boundary conditions (theorem3. 1, theorem3. 2 , theorem 3. 3 ) .

The results which obtained in chapters II and III concerning with the regularized sum of eigenfunctions of the first and second problem are considered modification of the previous studies .

# CHAPTER I
# PRELIMINARY DEFINITIONS AND THEOREMS

At the beginning we will give some basic definitions and theorems which will be used in our study.

## 1.1. Definitions and notions

Definition 1.1 : A function f (z) is analytic at a point $z_0$ if its derivative exists in a neighborhood of $z_0$ .

If a function is analytic at every point belonging to some region , we say that the function is analytic in the region.

Theorem 1.1 : The necessary and sufficient condition for a function $f(z) = u(x, y) + iv(x, y)$ to be analytic in a domain R is that its real and imaginary parts u and v are differentiable and have continuous first order partial derivative at (x, y ) and satisfy the Cauchy - Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$\forall (x,y)\in R$.

Examples :

1. The function $f(z) = z^2$ is differentiable in the whole z- plane, so f(z) is analytic in z-plane .

2. The function $f(z) = \text{Re}(z) = x$ is nowhere differentiable and $f(z)$ is not analytic .

3 . The function $f(z) = |z|^2$ is not analytic .

Definition 1.2 : A simple closed contour .

A contour C is called a simple closed contour if the initial and the final values of $f(z)$ are the same .

Definition 1. 3 : Simply connected domain R .

Is a domain such that every simple closed contour C within it contains only points of R, if R is not simply it is multiply - connected.

Definition 1. 4 : Singular points .

If f fails to be analytic at a point $z_0$ but f is analytic at some points in every neighborhood of $z_0$, then $z_0$ is called a singular point or singularity of f .

A singular point $z_0$ is said to be isolated if , in a ddition , there is some neighborhood of $z_0$ throughout which f is analytic except at $z_0$ .

Examples :

1.Let $f(z) = \frac{1}{z}$ , then $f'(z) = -\frac{1}{z^2}$ . $(z \neq 0)$

Hence f is analytic at every point except for $z = 0$ , where it is not even defined .

The point $z = 0$ is therefore a singular point . In fact it is an isolated singular point .

2. The function $f(z) = |z|^2$ has no singular points since it is nowhere analytic.

Definition 1. 5 : The residues

Let $z_0$ be an isolated singular point of a function $f(z)$ .

Then by the residues of $f(z)$ at $z_0$ , denoted by $\operatorname{Res}_{Z=Z_0} f(z)$ is meant the coefficient $C_{-1}$ in the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n , \qquad (1.1)$$

where

$$C_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad (n = 0 , \pm 1 , \pm 2, \dots )$$

Examples :

1 . Let $f(z) = \frac{1}{z(1-z)} = \frac{1}{z} + \frac{1}{1-z}$ , $0 < |z| < 1$ , $z_0 = 0$

it is easy to see that

$$C_{-K} = 0 , \forall K = 2 , 3 , 4 , \dots\dots ,$$

$$C_{-1} = C_0 = C_k = 1 , \forall K = 1, 2 , 3 , 4 , \dots$$

then

$$f(z) = \frac{1}{z} + 1 + z + z^2 + \dots..$$

So we have

$$\operatorname*{Res}_{z=0} f(z) = C_{-1} = 1$$

2 . Let $f(z) = \frac{e^z}{z^2}$ , $|z| > 0$, $z_0 = 0$

It is no difficult to calculate the coefficients

$$C_K \quad \forall K = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots,$$

$$\text{i.e. } C_{-2} = C_{-1} = C_0 = 1, \; C_{-K} = 0 \quad \forall K = 3, 4, 5, \ldots,$$

$$C_K = \frac{1}{(k+2)!} \quad \forall K = 1, 2, 3, \ldots .$$

Then

$$f(z) = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \frac{z^3}{5!} + \ldots$$

$$\operatorname*{Res}_{z=0} f(z) = C_{-1} = 1 .$$

3 . Let $f(z) = \frac{\sin z}{z^4}$ , $|z| > 0$, $z_0 = 0$

The Laurent expansion of f(z) can be written in the following formula

$$f(z) = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + \ldots .$$

Moreover

$$\operatorname*{Res}_{z=0} f(z) = C_{-1} = \frac{-1}{3!} = -\frac{1}{6}$$

Definition 1 . 6 : The poles .

Suppose the series (1.1) contains only a finite number of negative powers of $(z-z_0)$ . Let m be the highest power of $\frac{1}{(z-z_0)}$ appearing in ( 1.1 ) , so that

$$f(z) = \sum_{n=0}^{\infty} C_n \, (z - z_0)^n + \frac{C_{-1}}{(z-z_0)} + \frac{C_{-1}}{(z-z_0)} + \frac{C_{-2}}{(z-z_0)^2} +$$

$$\dots + \frac{C_{-m}}{(z-z_0)^m} ,$$

where

$$C_{-m} \neq 0.$$

The point $z_0$ is then called a pole of order m of f (z) . In particular , $z_0$ is said to be a simple if $m = 1$ , and we note that

$$\lim_{z \to z_0} f(z) = \infty$$

Examples :

1. $f(z) = \frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \dots , \quad |z| > 0$

then we have $z_0 = 0$ is a pole of order 2 .

2 . $f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + \dots$

so the point $z_0 = 0$ is a pole of order 3 .

3 . $f(z) = \dfrac{1}{z(1-z)} = \dfrac{1}{z} + 1 + z + z^2 + \ldots$

so the point $z_0 = 0$ is a simple pole of f (z) .

4 . $f(z) = \dfrac{1}{1+(z-1)} - \dfrac{1}{(z-1)}$ , $o < |z-1| < 1$

then

$$f(z) = -\frac{1}{(z-1)} + \sum_{n=0}^{\infty} (-1)^n (z-1)^n, \quad o < |z-1| < 1$$

$$\text{i.e.} f(z) = -\frac{1}{(z-1)} + 1 - (z-1) + (z-1)^2 \ldots$$

so the point $z_0 = 1$ is a simple pole of f (z)

Theorem 1. 2 : The Cauchy - Goursat theorem

If a function f is analytic at all points within and on a simple closed contour C , then

$$\int_C f(z)\, d z = 0$$

Theorem 1 . 3 : The residue theorem

Let C be a simple closed contour within and on which a function f is analytic except for a finite number of singular points $z_1, z_2, \ldots, z_n$ interior to C . If $B_1, B_2, \ldots, B_n$ denote the residue of f at those points , then

$$\int_C f(z)\, d z = 2 \pi i \ (B_1 + B_2 + \ldots + B_n),$$

where C is described in the positive sense .

Theorem 1.4 : Rouche's theorem .

Let f and g be functions which are analytic inside and on a positively oriented simple closed contour C .

If $|f(z)|>|g(z)|$ at each point z on C , then the functions f(z) and f (z) + g (z) have the same number of zeros , counting multiplicities , inside C .

Example :

1. Let $f(z) = a_0 z^n$ , $g(z) = a_1 z^{n-1} + a_2 z^{n-2} + .....+a_n$, $|z| = R$, $R>0$

$|f(z)| = |a_0| R^n$ , $|g(z)| = | a_1 z^{n-1} + ... +a_n | \leq | a_1 | R^{n-1} + ....+| a_n |$

$$\lim_{z\to\infty} \frac{|f(z)|}{|g(z)|} \geq \lim_{z\to\infty} \frac{|a_0| R^n}{|a_1| R^{n-1} + \ldots + |a_n|} = \infty$$

Then

$$\forall M > 0 \exists N > 0 : \frac{|f(z)|}{|g(z)|} > M \forall |z| > N$$

Put M = 1, then $|f(z)|>|g(z)|$ , so by theorem (1.4) we deduce that the functions f(z), f(z)+g(z) have the same number of zeros

i.e. $f(z) = a_0 z^n$ , $f(z) + g(z) = a_0 z^n + a_1 z^{n-1} + .... + a_n$

have n zeros in $|z| = R$ , and that is true by the fundamental theorem of algebra .

Remark : The residues at a pole can be calculated without making explicit use of a Laurent series by the following

formulae

(i) If $z_o$ is a simple pole of f (z) , then

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z - z_0)\, f(z) .$$

(ii) If $z_o$ is a pole of order m, m>1 of f (z) , then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

Examples :

1.Let $f(z) = \dfrac{1}{1+(z-1)} - \dfrac{1}{z-1}$ , $0 < |z-1| < 1$

from the definition ( 1.6) we deduce that $z_o=1$ is a simple pole of f (z) , then by remark (i) we have :

$$C_{-1} = \operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} (z - 1) \left[\frac{1}{1+(z-1)} - \frac{1}{(z-1)}\right]$$

$$= \lim_{z\to 1} \frac{z-1}{z} - \lim_{z\to 1} 1 = -1 .$$

Then

$$C_{-1} = -1$$

2 . The function $f(z) = \dfrac{e^z}{(z-1)^5}$ , $0 < |z-1|$

has the point $z_0 = 1$ a pole of order 5 , then by remark ( ii) we have

$$C_{-1} = \operatorname*{Res}_{z=1} = \frac{1}{4!} \lim_{z\to 1} \frac{d^4}{dz^4} \left[(z-1)^5 \frac{e^z}{(z-1)^5}\right]$$

$$= \frac{1}{4!} \lim_{z\to 1} \frac{d^4}{dz^4} e^z = \frac{e}{24}$$

Remark: For more detials and proof of theorems (1.1) -(1.4) see [23] , [24]

Definition 1.7 : Linear differential expression .

A linear differential expression means an expression of the form

$$l(y) = p_0(x)\, y^{(n)} + P_1(x)\, y^{(n-1)} + \dots + P_n(x)\, y . \qquad (1.2)$$

The functions $p_0(x)$ , $P_1(x)$, .... , $P_n(x)$ are called the coefficients and the number n is called the order of the differential expression .

We shall assume that the functions $\frac{1}{P_0(x)}$ , $P_1(x)$ , ...., $P_n(x)$ are continuous on a fixed, closed interval $[a, b]$ .

We denote by $C^{(n)}$ the set of all functions y (x) which have continuous derivatives up to the n th order inclusive in the interval $[a, b]$ .

Definition 1.8 : Boundary conditions .

We denote the values of the function y and its first ( n-1 ) successive derivatives at the boundary points a and b of the interval $[a, b]$ by

$$y_a, y'_a, y''_a, \dots, y_a^{(n-1)}, y_b, y'_b, y''_b, \dots, y_b^{(n-1)} \qquad (1.3)$$

let U (y) be a linear form in the variable ( 1. 3 ) , then

$$U(y) = \alpha_0 y_a + \alpha_1 y'_a + \dots + \alpha_{n-1} y_a^{(n-1)} + \beta_0 y_b + \dots + \beta_{n-1} y_b^{(n-1)} \qquad (1.4)$$

If several such forms $U_\gamma$ (y) have been specified , $\gamma = 1 , 2$ , ... , m , and if the conditions

$$U_\gamma(y) = 0 \ , \ \gamma = 1, 2, 3, \dots, m \qquad (1.5)$$

are imposed on the functions $y(x) \in C^{(n)}$, we call these the boundary conditions which the functions y must satisfy .

We denote by D the set of all functions $y \in C^{(n)}$ which satisfy a specifed system of boundary conditions of the form (1. 5) . Clearly, D is a linear subspace in $C^{(n)}$ which coincides with $C^{(n)}$ only if the conditions (1. 5) are entirely lacking or if all their coefficients vanish .

Suppose a certain differential expression $l(y)$ and a particular subspace D defined by conditions of the form (1. 5) are given. To each function $y \in D$, we let the function $u = l(y)$ correspond . This relation is a linear operator with D as its domain of definition , we denote it by L .

We write

$$u = L y \ .$$

The operator L is called the differential operator generated by the differential expression $l(y)$ and the boundary conditions (1. 5)

In this way - and this fact will be very important in the sequel - one and the same differential expression can generate various differential operators depending on the way in which the boundary conditions (1. 5) are chosen .

If , in particular , the conditions (1. 5) are absent, we obtain a differential operator , denoted by $L_1$ , with the domain of definition $D = C^{(n)}$ .

$L_1$ is obviously an extension of all other operators L which can be generated by the same differential expression l (y) .

If

$$U = \begin{vmatrix} U_1(y_1) & \cdots & U_1(y_n) \\ \cdots & \cdots & \cdots \\ U_m(y_1) & & U_m(y_n) \end{vmatrix}$$

Hence the following result :

$I^o$.If the rank of the matrix U is equal to r , the homogeneous, boundary - value problem has exactly ( n - r ) independent solutions .

In particular ,

$II^o$. (a) The homogeneous , bonndary - value problem has non - trivial solution if and only if the rank r of the matrix U is less than the order n of the differential expression l .

(b) For m < n , the homogeneous , bonndary - value problem always has a non- trivial solution .

(c) For m = n , the homogeneous , boundary - value problem has a non - trivial solution if and ond only if the determinant of the matrix U ( in this case , a square matrix ) vanishes .

Definition 1.9 : Eigenvalues and eigenfunctions .

A number $\lambda$ is called an eigenvalue of an operator L if there exists in the domain of definition of the operator L a function $y \equiv 0$ such that

$$Ly = \lambda y \tag{1. 6}$$

The function y is called the eigenfunction of the operator L for the eigenvalue $\lambda$ .

The operator L may be generated by the differential expression l (y) and the boundary conditions

$$U_1 (y) = 0 , .... , U_n (y) = 0 . \tag{1.7}$$

Since an eigenfunction y must belong to the domain of definition of the operator L , it must satisfy the conditions (1.5). Moreover, Ly = ly, and therefore (1.2) is equivalent to

$$l(y) = \lambda y \tag{1. 8}$$

Hence :the eigenvalues of an operator L are those values of the parameter $\lambda$ for which the homogeneous boundary - value problem

$$l (y) = \lambda y \ , U_\gamma (y) = 0 , \gamma = 1 , 2 , .... , m \tag{1. 9}$$

has non - trivial solutions , each of these non - trivial solutions is an eigenfunction belonging to $\lambda$ .

A linear combination of eigenfunctions which belong to one and the same eigenvalue $\lambda$ is itself an eigenfunction belonging to $\lambda$ . For if $Ly_1 = \lambda y_1$ and

$Ly_2 = \lambda y_2$, then

$L(c_1y_1+ c_2y_2) = \lambda (c_1y_1+ c_2y_2)$ for any constants $c_1$ and $c_2$ .

Since a homogeneous equation ( 1. 8) can have , for a given $\lambda$ not more than n linearly independent solutions , it follows that the aggregate of all eigenfunctions which belong to one and the same eigenvalue form a finite - dimensional space of dimension $\leq$ n . The dimension of this space is simply the number of linearly independent solutions of the boundary - value problem (1. 9) for the given eigenvalue $\lambda$, this number is called the multiplicity of the eigenvalue .

We shall specify conditions for the determination of eigenvalues .

Denoting by

$$\phi_1(x, \lambda), \phi_2(x, \lambda), \ldots\ldots, \phi_n(x, \lambda) \qquad (1.\ 10)$$

the fundamental system of solutions of the differential equation ( 1. 8) which satisfy the initial conditions

$$\phi_j^{(i-1)}(a,\lambda)=\begin{cases}0 & for \quad j\neq i\\ 1 & for \quad j=i\end{cases} \quad , j\ , i\ = 1\ , 2\ , \ldots\ , n\ .$$

(1.11)

From general theorems on the solutions of linear differential equations it follows that , for each fixed value of x in $[a, b]$, the functions (1. 10) are integral , analytic functions of the parameter $\lambda$ .

We have the following alternatives

I . For any differential operator L, only the following two possibilities can occur :

1 - Every number $\lambda$ is an eigenvalue for L .

2 - The operator has at most denumerably many eigenvalues ( in particular cases , none at all ) , and these eigenvalues can have no finite limit - point .

The case when m = n is of particular interest , and in the sequel we shall consider this particular case , if nothing to the contrary is stated .

We put

$$\Delta\ (\lambda) = \begin{vmatrix} U_1(\phi_1) & \cdots & U_1(\phi_n) \\ \cdots & \cdots & \cdots \\ U_n(\phi_1) & \cdots & U_n(\phi_n) \end{vmatrix} \qquad (1.\ 12)$$

By the preceding discussion, $\Delta(\lambda)$ is an integral, analytic function of $\lambda$ , the so - called characteristic determinant of the operator L.(or of the boundary - value problem Ly = 0 ), and the following theorems hold :

II . The eigenvalue of the operator L are the zeros of the function $\Delta\ (\lambda)$ . If $\Delta\ (\lambda)$ vanishes identically , then any number $\lambda$ is an eigenvalue of the oparator L .

If , however , $\Delta\ (\lambda)$ is not identically zero , the operator L has at most denumerably many eigenvalues , and these eigenvalues can have no finite limit-point .

If, in particular, the function $\Delta\ (\lambda)$ has no zeros at all, then the operator L has no eigenvalues.

An eigenvalue $\lambda$ may be a multiple zero of $\Delta\ (\lambda)$

III . If $\lambda_0$ is a zero of the charactristic determinant $\Delta\ (\lambda)$ with multiplicity $\gamma$, then the multiplicity of the eigenvalue $\lambda_0$ can not be greater than $\gamma$.

IV . If $\lambda_0$ is a simple zero of the charactristic determinant $\Delta\ (\lambda)$, then the multiplicity of the eigenvalue $\lambda_0$ of the operator L is also unity.

An eigenvalue is called simple if it is a simple zero of the charactristic determinant $\Delta\ (\lambda)$.

**1.2. The problem of inverting a differential operator.**

Let L be a differential operator, generated from the differential expression

$$l(y) = p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n(x)y,$$

and the boundary conditions

$$U_\gamma(y) = 0, \qquad \gamma = 1, 2, 3, \dots, n.$$

We assume that the homogeneous boundary value problem $Ly = 0$ has only the trivial solution $y = 0$,

i.e. for an arbitrary fundamental system of solutions $\phi_1$, $\phi_2, \dots \phi_n$ of the equation $l(y) = 0$, the rank of the matrix

$$\begin{vmatrix} U_1(\phi_1) & \dots & U_1(\phi_n) \\ \dots & \dots & \dots \\ U_n(\phi_1) & \dots & U_n(\phi_n) \end{vmatrix}$$

is equal to n , and so the determinant .

$\det|U_i(\phi_j)| \neq 0$ , i , j = 1 , 2 , .... n . (1. 13)

By II L then has an inverse $L^{-1}$ , whose domain of definition coincides with the range of values of the operator L. We set ourselves the task of finding an explicit expression for $L^{-1}$. As we shall see , $L^{-1}$ turns out to be an integral operator with a continuous Kernel , the kernel is called *Green's* function for the operator L .

We shall give below an explicit expression for the *Green's* function .

**1.3. Construction of *Green's* function .**

*Green's* function for an operator L is to be understood to be a function G ( x,$\xi$) satisfying the following conditions:

1 - G(x,$\xi$ ) is continuous and has continuous derivatives with respect to x up to order (n-2 ) inclusive for all values of x and $\xi$ in the interval $[a,b]$ .

2 . For any fixed value of $\xi$ in the interval (a,b) the function G(x , $\xi$ ) has continuous derivatives of orders ( n-1) and n with respect to x in each of the intervals $[a,\xi)$ and $(\xi , b]$, the ( n-1) th derivative is discontinuous at x = $\xi$ with a jump of

$$\frac{1}{p_0(\xi)}:$$

$$\frac{\partial^{n-1}}{\partial x^{n-1}} G(\xi+0,\xi) - \frac{\partial^{n-1}}{\partial x^{n-1}} G(\xi-0,\xi) = \frac{1}{p_0(\xi)}$$

3 . In each of the intervals $[a, \xi)$ and $(\xi, b]$, $G(x,\xi)$, considered as a function of x, satisfies the equation $l(G) = 0$ and the boundary conditions $U_\gamma(G) = 0$, $\gamma = 1, 2, ..., n$.

Theorem 1.5 : If the boundary - value problem $L y = 0$ has only the trivial solution, then the operator L has one and only one *Green's* function .

Proof : see [ 25 ]

**1.4 . Inversion of a differential operator by means of the *Green's* function .**

Suppose that the equation $L y = 0$ has only the trivial solution $y = 0$, so that the inverse $L^{-1}$ and its *Green's* function exist . if $L^{-1} f = y$, then $L y = f$ (1. 14)

i.e. y is a solution of the equation

$$l(y) = f, \qquad (1.\ 15)$$

and also satisfies the boundary conditions

$$U_\gamma(y) = 0, \quad \gamma = 1, 2, ...., n. \qquad (1.\ 16)$$

We shall show that this solution exists for any function f(x) which is continuous in the interval $[a,b]$, and that it can be determined by means of the *Green's* function . More precisely , the following theorem is valid .

Theorem 1. 6 :

If the equation $L y = 0$ has only the trivial solution, then, for any function f (x) which is continuous in the interval

$[a, b]$ , there exists a solution of the equation $Ly = f$, this solution is expressed by the formula

$$y(x) = \int_a^b G(x,\xi) f(\xi) d\xi \qquad (1.17)$$

where $G(x, \xi)$ denotes the *Green's* function for the operator $L$.

Proof : see [25]

**1.5.** ***Green's*** **function for the operator L - $\lambda$ I .**

Let L be an operator generated by the expression $l(y)$ and the conditions

$U_\gamma(y) = 0$ , $\gamma = 1, 2, \ldots, n$. The expression for the *Green's* function of the operator $L - \lambda I$ is given by the formula

$$G(x, \xi, \lambda) = \frac{(-1)^n}{\Delta(\lambda)} H(x, \xi, \lambda), \qquad (1.18)$$

where

$$\Delta(\lambda) = \begin{vmatrix} U_1(\phi_1) & U_1(\phi_2) & \ldots & U_1(\phi_n) \\ \ldots & \ldots & & \ldots \\ U_n(\phi_1) & U_n(\phi_2) & \ldots & U_n(\phi_n) \end{vmatrix}, \qquad (1.19)$$

$$H(x, \xi, \lambda) = \begin{vmatrix} \phi_1(x,\lambda) \ldots & \phi_n(x,\lambda) & g(x,\xi) \\ U_1(\phi_1) \ldots & U_1(\phi_n) & U_1(g) \\ \ldots & \ldots & \ldots \\ U_n(\phi_1) \ldots & U_n(\phi_n) & U_n(g) \end{vmatrix}, \qquad (1.20)$$

$$g(x,\xi) = \pm \frac{1}{2W(\xi)} \begin{vmatrix} \phi_1(x,\lambda) & \cdots & \phi_n(x,\lambda) \\ \phi_1^{n-2}(\xi,\lambda) & \cdots & \phi_n^{n-2}(\xi,\lambda) \\ \cdots & \cdots & \cdots \\ \phi_1(\xi,\lambda) & \cdots & \phi_n(\xi,\lambda) \end{vmatrix} \qquad (1.21)$$

( The positive sign being taken if $x > \xi$ , and the negative sign if $x < \xi$ ),

and

$$W(\xi) = \begin{vmatrix} \phi_1^{n-1}(\xi,\lambda) & \cdots & \phi_n^{n-1}(\xi,\lambda) \\ \cdots & \cdots & \cdots \\ \phi_1(\xi,\lambda) & \cdots & \phi_n(\xi,\lambda) \end{vmatrix} \qquad (1.22)$$

**1.6. Rieman*n's* Zeta Functions $\zeta$ (s) and $\zeta$ ( s , a )**

Definition 1.10 : Let $s = \sigma + it$ where $\sigma$ and t are real , then ; if $\delta > 0$ , the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (1.23)$$

is a uniformly convergent series of analytic functions in any domain in which $\sigma \geq 1 + \delta$ ; and consequently the series is an analytic function of s in such a domain . The function is called the Rieman*n's* Zeta function .

Many of the properties possessed by the Rieman*n's* Zeta - function are particular cases of properties possessed by a more general function defined, when $\sigma \geq 1 + \delta$ , by the equation

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(a+n)^s} \qquad (1.24)$$

where a is a constant . For simplicity , we shall suppose that $0 < a \leq 1$ , and then we take $\arg(a+n) = 0$ . It is evident that $\zeta(s,1) = \zeta(s)$ . The function $\zeta(s,a)$ is called the generalized Riemann's Zeta function

Definition(1.11).The numbers$B_n$, representing the coefficients of $\frac{t^n}{n!}$ in the expansion of the function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} ; \qquad 0 < |t| < 2\pi \tag{1. 25}$$

are called Bernoulli numbers . Thus the function $\frac{t}{e^t - 1}$ is a generating function for the Bernoulli numbers .

We note that the Bernoulli numbers has the following properties :

1 - $B_n = \sum_{k=0}^{n} \binom{n}{k} B_k$ ; $B_0 = 1$ , $(n \neq 1)$

2 - All the Bernoulli numbers with odd index are equal to zero except that $B_1 = -\frac{1}{2}$ ; that is , $B_{2n+1} = 0$ for n a natural number .

3 - For some values of $B_n$ , we have

$B_0 = 1$ , $B_1 = -\frac{1}{2}$ , $B_2 = \frac{1}{6}$ , $B_4 = -\frac{1}{30}$ ,

$B_6 = \frac{1}{42}$ , $B_8 = -\frac{1}{30}$ , $B_{10} = \frac{5}{60}$ , $B_{12} = \frac{691}{2730}$, ....

Definition (1. 12) . The Bernoulli polynomials $B_n(x)$ are defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k \, x^{n-k} . \tag{1. 26}$$

We can also define the Bernoulli polynomials as the following

Definition (1. 13) The Bernoulli polynojmials $B_n(x)$ ,

representing the coefficients of $\frac{t^{n-1}}{n!}$ in the expansion of the function

$$\frac{e^{xt}}{e^{t}-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^{n-1}}{n!}, \qquad [\, 0<|t|<2\pi \,]. \quad (1.27)$$

From (1. 27), we can obtian the following polynomials:

$B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$,

$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$, $B_5(x) = x^5 - \frac{5}{2}x^2 + \frac{5}{3}x^3 - \frac{1}{6}x, \ldots$

If we put $x = 0$ in both sides of (1. 27) , we have

$$\frac{1}{e^{t}-1} = \sum_{n=0}^{\infty} B_n(0) \frac{t^{n-1}}{n!}, \qquad [\, 0<|t|<2\pi \,]. \quad (1.28)$$

Upon using formula ( 1. 25) and (1. 28) , we get

$B_n(0) = B_n$, $B_1(1) = - B_1 = \frac{1}{2}$, $B_n(1) = B_n$, $(n \neq 1)$

Riemann's Zeta - functions $\zeta(s)$ and $\zeta(s, a)$ has the following properties

(1) $\zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} e^{-ax}}{1-e^{-x}}\, dx$, (1. 29)

where $\sigma \geq 1 + \delta$, $\arg x = 0$

(2) $\lim_{s\to 1} \frac{\zeta(s,a)}{\Gamma(1-s)} = -1$

(3) $\zeta(-m, a) = -\frac{B_{m+1}(a)}{m+1}$

(4) $\zeta(-2m) = 0, \zeta(1-2m) = (-1)^m B_m/(2m)$

$= - B_{2m}/(2m)$ ; $(m = 1, 2, 3, \ldots)$

$\zeta(0) = -\frac{1}{2}$

(5) $\zeta(2m) = \frac{2^{2m-1}\pi^m |B_{2m}|}{(2m)!}$, (6) $\zeta'(0) = -\frac{1}{2}\ln 2\pi$

Remark : for more detials see [27] , [29] .

# CHAPTER II
# REGULARIZED TRACES AND SUMS OF EIGENFUNCTIONS FOR NON - PERIODIC BOUNDARY CONDITIONS

## 2.1 . INTRODUCTION.

For large values of $|\lambda|$ , approximation formulae and , indeed, a symptotic formlae can be given for the eigenvalues and eigenfunctions of a differential operator . Such formulae are not only of interest in themselves , but they also find application at a decisive point in the proof of certain theorems in the theory of differential operators, particularly finding the regularized traces and sums of eigenfunctions for the differential operators .

It turns out that the behaviour of the eigenvalues $\lambda_\gamma$ and eigenfunctions for an arbitrary differential operator as $|\lambda_\gamma|$ increases can , to a first approximation, be characterized by the eigenvalues and eigenfunctions of the operator which is generated by the same boundary conditions but the simplest possible differential expression of the fourth order $l(y) = y^{(4)}$.

We shall first investigate the asymptotic behaviour of solutions of $l(y) = \lambda y$ for large $|\lambda|$, where $l(y)$ is a linear differential expression in the form.

$$l(y) = y^{(4)}(x) + q(x)\, y, \quad 0 \le x \le \pi \qquad (2.1)$$

and the boundary conditions

$$\begin{aligned} U_1(y) &= y(0) = 0, \\ U_2(y) &= y''(0) = 0, \\ U_3(y) &= y(\pi) = 0, \\ U_4(y) &= y''(\pi) = 0, \end{aligned} \qquad (2.2)$$

First we put $\lambda = \rho^4$, then the equation $l(y) = \lambda y$ takes the form

$$l(y) = \rho^4 y \qquad (2.3)$$

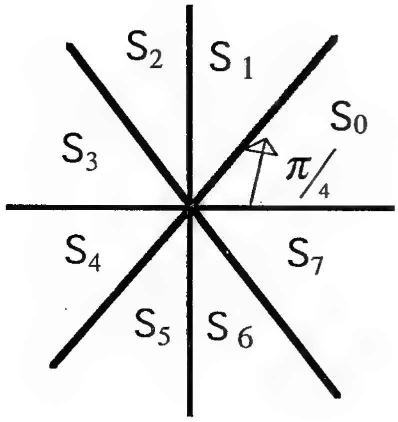
or, in more detail

$$y^{(4)}(x) + q(x)\, y = \rho^4 y \qquad (2.4)$$

**2.2. The domains S and T.**

We divide the complex $\rho$ - Plane into eight sectors $S_k$, $k = 0, 1, 2, \ldots, 7$, defined by

$$\frac{k\pi}{4} \le \arg \rho \le \frac{(k+1)\pi}{4}. \qquad (2.5)$$




(Fig . 1 )

As we shall see later , the asymptotic formulae for the solution y of the equation (2. 4) depend essentially on which sector $S_k$ the point $\rho$ lies in . We denote by

$$w_1 , w_2 , w_3 , w_4 ,$$

the different $4^{th}$ roots of 1 arranged in an order such that

$$\mathrm{Re}(\rho w_1) \leq \mathrm{Re}(\rho w_2) \leq \mathrm{Re}(\rho w_3) \leq \mathrm{Re}(\rho w_4) \qquad (2.6)$$

$\forall \rho \in S_k$ ,where Re (z) means the real part of z .

For example if $\rho \in S_0$ , then $w_1 = -1$, $w_2 = i$, $w_3 = -i$, $w_4 = 1$.

We consider more general domains , namely those which are obtained from the sectors $S_k$ by a translation $\rho \rightarrow \rho + c$ , where c is a fixed complex number . These new sectors with their vertices at the point $\rho = -c$ will correspondingly be denoted by $T_k$ , $k = 0, 1, 2, \ldots, 7$ .

We see that , for $\rho \in T_k$ , the inequalities

$$\mathrm{Re}((\rho + c)w_1) \leq \mathrm{Re}((\rho + c)w_2) \leq \mathrm{Re}((\rho + c)w_3) \leq \mathrm{Re}((\rho + c)w_4), \qquad (2.7)$$

hold , for a suitable ordering of the numbers $w_1$ , $w_2$ , $w_3$ , and $w_4$ . In the sequel we shall let $\rho$ vary in a fixed domain and so we shall write simply S and T instead $S_k$ and $T_k$ .

**2.3.The asymptotic formulae for the fundamental solutions of (2.4).**

Let $\phi_1(x,\rho)$ , $\phi_2(x,\rho)$ , $\phi_3(x,\rho)$ and $\phi_4(x,\rho)$ denote a fundamental system of solutions of the differential equation (2. 4 ) which satisfy the initial conditions

$$\phi_j^{(i-1)}(0, \rho) = \delta_{ji} = \begin{cases} 0 & for \quad j \neq i \\ 1 & for \quad j = i \end{cases} ; j,i = 1,2,3,4 \qquad (2.\ 8)$$

where $\delta_{ji}$ Kronecker delta .

For the functions $\phi_k(x, \rho)$ ( k = 1 , 2 ,3 ,4 ) , we have the following theorem

**Theorem 2 . 1** : The asymptotic formulae for the fundamental system of solutions of the differential equation (2. 4) are given by

$$\phi_1(x,\rho) = \frac{1}{2}(\cos\rho x + \cosh\rho x) + \frac{1}{8\rho_3}(\sin\rho x - \sinh\rho x)$$
$$\int_0^x q(t)dt + \frac{3}{16\rho^4}(\cos\rho x + \cosh\rho x)(q(x) - q(0)) + \ldots$$

$$\phi_2(x,\rho) = \frac{1}{2\rho}(\sin\rho x + \sinh\rho x) - \frac{1}{8\rho^4}(\cos\rho x + \cosh\rho x)\int_0^x q(t)dt + \frac{1}{16\rho^5}(\sin\rho x + \sinh\rho x)(3q(x) - q(0)) + \ldots$$

$$\phi_3(x,\rho) = \frac{1}{2\rho^2}(\cosh\rho x - \cos\rho x) - \frac{1}{8\rho^5}(\sin\rho x + \sinh\rho x)\int_0^x q(t)dt + \frac{1}{16\rho^6}(\cosh\rho x - \cos\rho x)(3q(x)+q(0)) + \ldots$$

and

$$\phi_4(x,\rho) = \frac{1}{2\rho^3}(\sinh\rho x - \sin\rho x) + \frac{1}{8\rho^6}(\cos\rho x - \cosh\rho x)\int_0^x q(t)dt + \frac{3}{16\rho^7}(\sinh\rho x - \sin\rho x)(q(x)+q(0)) + \ldots$$

(2. 9)

Proof : Puting $q(x) \equiv 0$ in (2. 4) , we have the differential equation

$$y^{(4)}(x) = \rho^4 y(x) . \qquad (2.\ 10)$$

The general solution of (2. 10) is given by the following formula

$$y(x, \rho) = c_1 \sin\rho x + c_2 \sinh\rho x + c_3 \cos\rho x + c_4 \cosh\rho x , \qquad (2.\ 11)$$

where $c_i$ ( $i = 1, 2, 3, 4$ ) are constants .

If $\phi_1^{(0)}(x,\rho), \phi_2^{(0)}(x,\rho), \phi_3^{(0)}(x,\rho)$ and $\phi_4^{(0)}(x,\rho)$ are fundamental solutions of ( 2. 10) , we have

$$\phi_1^{(0)}(x,\rho) = \frac{1}{2}(\cos\rho x + \cosh\rho x)$$

$$\phi_2^{(0)}(x,\rho)=\frac{1}{2\rho}\,(\sin\rho x+\sinh\rho x)$$

$$\phi_3^{(0)}(x,\rho)=\frac{1}{2\rho^2}\,(\cosh\rho x-\cos\rho x)$$

and

$$\phi_4^{(0)}(x,\rho)=\frac{1}{2\rho^3}\,(\sinh\rho x-\sin\rho x) \tag{2. 12}$$

To solve the nonhomogeneous differential equation (2.4) ($q(x)\ \equiv 0$ ), we use the method of variation of parameters namely , we put

$$\phi_p(x,\rho)= C_1(x,\rho)\sin\rho x+ C_2(x,\rho)\sinh\rho x+ C_3(x,\rho)\cos\rho x+C_4(x,\rho)\cosh\rho x \tag{2. 13}$$

For the functions $C_k$ ( $x,\rho$ ) , we have the following system

$$C'_1(x,\rho)\sin\rho x+C'_2(x,\rho)\sinh\rho x+C'_3(x,\rho)\cos\rho x+C'_4(x,\rho)\cosh\rho x=0,$$

$$C'_1(x,\rho)\cos\rho x+C'_2(x,\rho)\cosh\rho x-C'_3(x,\rho)\sin\rho x+C'_4(x,\rho)\sinh\rho x=0,$$

$$-C'_1(x,\rho)\sin\rho x+C'_2(x,\rho)\sinh\rho x-C'_3(x,\rho)\cos\rho x+C'_4(x,\rho)\cosh\rho x=0,$$

and

$$-C'_1(x,\rho)\cos\rho x+C'_2(x,\rho)\cosh\rho x+C'_3(x,\rho)\sin\rho x+C'_4(x,\rho)\sinh\rho x=-\frac{1}{\rho^3}\,q(x)\,\phi(x,\rho) \tag{2. 14}$$

Solving the system (2. 14) , we get

$$\left.\begin{aligned}
C_1(x, \rho) &= \frac{1}{2\rho^3} \int_0^x q(t) \cos\rho t \, \phi(t, \rho) \, dt, \\
C_2(x, \rho) &= -\frac{1}{2\rho^3} \int_0^x q(t) \cosh\rho t \, \phi(t, \rho) \, dt, \\
C_3(x, \rho) &= -\frac{1}{2\rho^3} \int_0^x q(t) \sin\rho t \, \phi(t, \rho) \, dt,
\end{aligned}\right] \qquad (2.15)$$

and

$$C_4(x, \rho) = \frac{1}{2\rho^3} \int_0^x q(t) \sinh\rho t \, \phi(t, \rho) \, dt,$$

Substituting (2.15) in (2. 13) and using the conditions (2. 8), we have the following integral equations

$$\phi_1(x, \rho) = \frac{1}{2}(\cos\rho x + \cosh\rho x) + \frac{1}{2\rho^3} \int_0^x q(t)\phi_1(t, \rho) \left[\sin\rho(x-t) - \sinh\rho(x-t)\right] dt,$$

$$\phi_2(x, \rho) = \frac{1}{2\rho}(\sin\rho x + \sinh\rho x) + \frac{1}{2\rho^3} \int_0^x q(t)\phi_2(t, \rho) \left[\sin\rho(x-t) - \sinh\rho(x-t)\right] dt,$$

$$\phi_3(x, \rho) = \frac{1}{2\rho^2}(\cosh\rho x - \cos\rho x) + \frac{1}{2\rho^3} \int_0^x q(t)\phi_3(t, \rho) \left[\sin\rho(x-t) - \sinh\rho(x-t)\right] dt,$$

and

$$\phi_4(x, \rho) = \frac{1}{2\rho^3}(\sinh\rho x - \sin\rho x) + \frac{1}{2\rho^3} \int_0^x q(t)\phi_4(t, \rho) \left[\sin\rho(x-t) - \sinh\rho(x-t)\right] dt \qquad (2.16)$$

Upon using the successive approximation method , we can put ( 2. 16) in the following form

$$\phi_{k,n}(x, \rho) = \phi_k^{(0)}(x, \rho) + \frac{1}{2\rho^3}\int_0^x q(t)\phi_{k,n-1}(t, \rho)$$
$$[\sin\rho(x-t) - \sinh\rho(x-t)]\,dt, \qquad (2.\ 17)$$

where $k = 1, 2, 3, 4$, and $n = 1, 2, 3, \ldots$

By straight forward calculation, using (2. 12) and (2.17), we get the asymptotic formulae ( 2. 9 ) for the fundamental solutions of ( 2. 4 ) . This proves the theorem(2 . 1 )

**Lemma 2. 1.** The asymptotic formulae for the derivative for the fundamental system of solutions of the differential equation ( 2. 4 ) are given by the following forms

$$\phi'_1(x,\rho) = \frac{\rho}{2}(\sinh\rho x - \sin\rho x) + \frac{1}{8\rho^2}(\cos\rho x - \cosh\rho x)$$
$$\int_0^x q(t)\,dt + \frac{1}{16\rho^3}(\sinh\rho x - \sin\rho x)(q(x) - 3q(0)) + \ldots,$$

$$\phi'_2(x,\rho) = \frac{1}{2}(\cos\rho x + \cosh\rho x) + \frac{1}{8\rho^3}(\sin\rho x - \sinh\rho x)$$
$$\int_0^x q(t)\,dt + \frac{1}{16\rho^4}(\cos\rho x + \cosh\rho x)(q(x) - q(0) + \ldots,$$

$$\phi'_3(x,\rho) = \frac{1}{2\rho}(\sin\rho x + \sinh\rho x) - \frac{1}{8\rho^4}(\cos\rho x + \cosh\rho x)$$

$$\int_0^x q(t)\,dt + \frac{1}{16\rho^5}(\sinh\rho x + \sin\rho x)(q(x) + q(0) + \dots,$$

and

$$\phi'_4(x,\rho) = \frac{1}{2\rho^2}(\cosh\rho x - \cos\rho x) - \frac{1}{8\rho^5}(\sin\rho x + \sinh\rho x)$$

$$\int_0^x q(t)\,dt + \frac{1}{16\rho^6}(\cosh\rho x - \cos\rho x)(q(x) + 3\,q(0)) + \dots$$

(2. 18)

Proof : Differentiate (2. 17 ) with respect to x , we have

$$\phi'_{k,n}(x,\rho) = \phi'^{(0)}_k(x,\rho) + \frac{1}{2\rho^2}\int_0^x q(t)\phi_{k,n-1}(t,\rho)\left[\cos\rho(x-t) - \cosh\rho(x-t)\right]dt$$

(2. 19)

Solving the integral equations ( 2. 19) by using the successive approximation method and the formula ( 2. 12 ) , we get (2. 18) .

Proceeding now in a manner similar to before , it is not difficult to obtain the asymptotic formulae for $\phi''_k(x, \rho)$ and $\phi'''_k(x, \rho)$ as in the following forms

$$\phi''_1(x, \rho) = \frac{\rho^2}{2}(\cosh\rho x - \cos\rho x) - \frac{1}{8\rho}(\sin\rho x + \sinh\rho x)$$

$$\int_0^x q(t)\,dt + \frac{1}{16\rho^2}(\cos\rho x - \cosh\rho x)(q(x) + 3\,q(0)) + \dots,$$

$$\phi''_2(x,\rho)=\frac{\rho}{2}(\sinh\rho x-\sin\rho x)+\frac{1}{8\rho^{2}}(\cos\rho x-\cosh\rho x)$$

$$\int_0^x q(t)\,dt+\frac{1}{16\rho^{3}}(\sin\rho x-\sinh\rho x)(q(x)+q(0))+\ldots,$$

$$\phi''_3(x,\rho)=\frac{1}{2}(\cos\rho x+\cosh\rho x)+\frac{1}{8\rho^{3}}(\sin\rho x-\sinh\rho x)$$

$$\int_0^x q(t)dt+\frac{1}{16\rho^{4}}(\cos\rho x+\cosh\rho x)(q(0)-q(x))+\ldots,$$

and

$$\phi''_4(x,\rho)=\frac{1}{2\rho}(\sinh\rho x+\sin\rho x)-\frac{1}{8\rho^{4}}(\cos\rho x+\cosh\rho x)$$

$$\int_0^x q(t)\,dt+\frac{1}{16\rho^{5}}(\sinh\rho x+\sin\rho x)(3q(0)-q(x))+\ldots, \tag{2. 20}$$

For the functions $\phi'''_k(x,\rho)$ ( $k=1,2,3,4$ ), we get the following asymptotic formulae :

$$\phi'''_1(x,\rho)=\frac{\rho^3}{2}(\sin\rho x+\sinh\rho x)-\frac{1}{8}(\cos\rho x+\cosh\rho x)$$

$$\int_0^x q(t)\,dt-\frac{3}{16\rho}(\sin\rho x+\sinh\rho x)(q(x)+q(0))+\ldots,$$

$$\phi'''_2(x,\rho)=\frac{\rho^2}{2}(\cosh\rho x-\cos\rho x)-\frac{1}{8\rho}(\sin\rho x+\sinh\rho x)$$

$$\int_0^x q(t)dt+\frac{1}{16\rho^{2}}(\cos\rho x-\cosh\rho x)(3q(x)+q(0))+\ldots,$$

$$\phi'''_3(x,\rho)=\frac{\rho^2}{2}(\sinh\rho x-\sin\rho x)+\frac{1}{8\rho^2}(\cos\rho x-\cosh\rho x)$$

$$\int_0^x q(t)\,dt+\frac{1}{16\rho^3}(\sinh\rho x-\sin\rho x)\,(q(0)-3q(x))+\ldots,$$

and

$$\phi'''_4(x,\rho)=\frac{1}{2}(\cos\rho x+\cosh\rho x)+\frac{1}{8\rho^3}(\sin\rho x-\sinh\rho x)$$

$$\int_0^x q(t)\,dt+\frac{3}{16\rho^4}(\cos\rho x+\cosh\rho x)(q(0)-q(x))+\ldots, \tag{2. 21}$$

**2.4 .Asymptotic behaviour of the eigenvalues of problem(2.1) -(2.2)**

The result of section 3 enable us to give asymptotic approximation for the behaviour of the sequence of eigenvalues at infinity .

**Theorem 2 . 2** : A differential operator of the fourth order which is generated by an expression ( 2. 1 ) and boundary conditions (2.2) has precisely denumerably many eigenvalues, whose behaviour at infinity is specified by the following formulae

(1) If $\rho\in S_7\cup S_0$ , then

$$\rho_k\sim k\left[1+\frac{1}{4\pi k^4}\int_0^\pi q(t)dt+O(\frac{1}{k^5})\right] \tag{2. 22}$$

(2) If $\rho\in S_1\cup S_2$, then

$$\rho_k\sim ik\left[1+\frac{1}{4\pi k^4}\int_0^\pi q(t)dt+O(\frac{1}{k^5})\right] \tag{2. 23}$$

(3) If $\rho \in S_3 \cup S_4$, then

$$\rho_k \sim -k\left[1+\frac{1}{4\pi k^4}\int_0^{\pi} q(t)dt + O\left(\frac{1}{k^5}\right)\right] \qquad (2.24)$$

(4) If $\rho \in S_5 \cup S_6$, then

$$\rho_k \sim -ki\left[1+\frac{1}{4\pi k^4}\int_0^{\pi} q(t)dt + O\left(\frac{1}{k^5}\right)\right] \qquad (2.25)$$

Proof : Upon using formula (1. 12 ) in chapter I , we have

$$\Delta(\rho)=\begin{vmatrix} U_1(\phi_1) & U_1(\phi_2) & U_1(\phi_3) & U_1(\phi_4) \\ U_2(\phi_1) & U_2(\phi_2) & U_2(\phi_3) & U_2(\phi_4) \\ U_3(\phi_1) & U_3(\phi_2) & U_3(\phi_3) & U_3(\phi_4) \\ U_4(\phi_1) & U_4(\phi_2) & U_4(\phi_3) & U_4(\phi_4) \end{vmatrix} \qquad (2.26)$$

But from (2. 2 ) and the conditions (2. 8) , we get

$U_1(\phi_1) = \phi_1(0, \rho) = 1$, $U_1(\phi_2) = \phi_2(0, \rho) = 0$,

$U_1(\phi_3) = \phi_3(0, \rho) = 0$, $U_1(\phi_4) = \phi_4(0, \rho) = 0$,

$U_2(\phi_1) = \phi''_1(0, \rho) = 0$, $U_2(\phi_2) = \phi''_2(0, \rho) = 0$,

$U_2(\phi_3) = \phi''_3(0, \rho) = 1$, $U_2(\phi_4) = \phi''_4(0, \rho) = 0$

(2. 27)

Substituting ( 2. 27 ) in (2. 26 ) , then

$$\Delta(\rho)=\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \phi_1(\pi,\rho) & \phi_2(\pi,\rho) & \phi_3(\pi,\rho) & \phi_4(\pi,\rho) \\ \phi''_1(\pi,\rho) & \phi''_2(\pi,\rho) & \phi''_3(\pi,\rho) & \phi''_4(\pi,\rho) \end{vmatrix}$$

$$= \phi_4(\pi,\rho)\ \phi''_2(\pi,\rho) - \phi_2(\pi,\rho)\ \phi''_4(\pi,\rho) \qquad (2.\ 28)$$

Now inserting the values of the functions $\phi_2(\pi,\rho)$ , $\phi_4(\pi,\rho)$, $\phi''_2(\pi,\rho)$ and $\phi''_4(\pi,\rho)$ from the formulae(2. 9) and (2. 20) into ( 2. 28) , we obtain the following equation for the determination of the eigenvalues

$$-\frac{\sin\rho\pi\ \sinh\rho\pi}{\rho^2} + \frac{1}{4\rho^5}(\sinh\rho\pi\ \cos\rho\pi + \sin\rho\pi\ \cosh\rho\pi) \int_0^\pi q(t)dt - \frac{\sin\rho\pi\ \sinh\rho\pi}{4\rho^6}(q(0)+q(\pi)) - \frac{1}{16\rho^8}\cos\rho\pi\ \cosh\rho\pi \left(\int_0^\pi q(t)dt\right)^2 + O\left(\frac{1}{\rho^{10}}\right)=0 \qquad (2.\ 29)$$

The equation (2. 29 ) can be written in the form :

$$\sum_{k=1}^{4} e^{\alpha_k \pi\rho}\left\{\frac{C_1^{(k)}}{\rho^2}+\frac{C_2^{(k)}}{\rho^5}+\frac{C_3^{(k)}}{\rho^6}+\dots\dots\right\}=0\ , \qquad (2.\ 30)$$

where

$$\alpha_1 = 1+i\ ,\alpha_2 = -1+i\ ,\ \alpha_3 = -1-i\ ,\ \alpha_4 = 1-i\ ,$$

$$C_1^{(1)}=-\frac{1}{4i},C_1^{(2)}=\frac{1}{4i},C_1^{(3)}=-\frac{1}{4i},C_1^{(4)}=\frac{1}{4i},$$

$$c_2^{(1)}=\frac{1}{16}(1-i)\int_o^{\pi} q(t)dt,\ c_2^{(2)}=-\frac{1}{16}(1+i)\int_o^{\pi} q(t)dt,$$
$$c_2^{(3)}=-\frac{1}{16}(1-i)\int_o^{\pi} q(t)dt,\ c_2^{(4)}=\frac{1}{16}(1+i)\int_o^{\pi} q(t)dt,$$
$$c_3^{(1)}=-\frac{1}{16i}\ (q(0)+q(\pi)),\ c_3^{(2)}=\frac{1}{16i}\ (q(0)+q(\pi)),$$
$$c_3^{(3)}=-\frac{1}{16i}\ (q(0)+q(\pi)),\ c_3^{(4)}=\frac{1}{16i}\ (q(0)+q(\pi)),.... \quad (2.\ 31)$$

Using formula (2. 30) , we shall obtain an asymptotic formulae for the eigenvalues in the following sectors :

(1) If $\rho\in\ S_7\cup S_0$ , then (2. 30) takes the form

$$e^{\alpha_1\pi\rho}\left\{\frac{C_1^{(1)}}{\rho^2}+\frac{C_2^{(1)}}{\rho^5}+\frac{C_3^{(1)}}{\rho^6}+........\right\}+\ e^{\alpha_4\pi\rho}\left\{\frac{C_1^{(4)}}{\rho^2}+\frac{C_2^{(4)}}{\rho^5}+\frac{C_3^{(4)}}{\rho^6}+........\right\}=0 \quad (2.\ 32)$$

Formula (2. 32) can be put in the following form :

$$e^{(\alpha_1-\alpha_4)\pi\rho}=-\frac{c_1^{(4)}+\dfrac{c_2^{(4)}}{\rho 3}+\dfrac{c_3^{(4)}}{\rho 4}+........}{c_1^{(1)}+\dfrac{c_2^{(1)}}{\rho 3}+\dfrac{c_3^{(1)}}{\rho 4}+........}$$

$$(\alpha_1-\alpha_4)\ \pi\ \ \rho=2\ \mathrm{k}\ \pi\ \mathrm{i}+\ln\ (1+\frac{c_2^{(4)}}{c_1^{(4)}\rho^3}+\frac{c_3^{(4)}}{c_1^{(4)}\rho^4}+....)$$
$$-\ \ln\ \ (1+\frac{c_2^{(1)}}{c_1^{(1)}\rho^3}+\frac{c_3^{(1)}}{c_1^{(1)}\rho^4}+....\ )\ ,$$

using formulae (2. 31 ) , we have

$$2\mathrm{i}\ \rho\,\pi\ =\ 2\mathrm{ik}\pi+\frac{c_2^{(4)}}{c_1^{(4)}\rho^3}+\frac{c_3^{(4)}}{c_1^{(4)}\rho^4}+.....-\frac{c_2^{(1)}}{c_1^{(1)}\rho^3}-\frac{c_3^{(1)}}{c_1^{(1)}\rho^4}-...$$
$$=\ 2\mathrm{ik}\pi+\frac{1}{\rho^3}\left(\frac{c_2^{(4)}}{c_1^{(4)}}-\frac{c_2^{(1)}}{c_1^{(1)}}\right)+\frac{1}{\rho^4}\left(\frac{c_3^{(4)}}{c_1^{(4)}}-\frac{c_3^{(1)}}{c_1^{(1)}}\right)+...$$

$$2i\rho\pi = 2ik\pi + \frac{i}{2\rho^{3}} \int_{o}^{\pi} q\,(t)\,dt + ...$$

$$\rho_k \sim k + \frac{1}{4\pi k^{3}} \int_{o}^{\pi} q\,(t)\,dt + ...$$

$$\rho_k \sim k\left[1 + \frac{1}{4\pi k^{4}} \int_{0}^{\pi} q(t)dt + .......\right]$$

(2) If $\rho \in S_1 \cup S_2$ , then (2. 30) can be written in the form

$$e^{\alpha_1 \pi\rho} \left\{ \frac{c_1^{(1)}}{\rho^{2}} + \frac{c_2^{(1)}}{\rho^{5}} + \frac{c_3^{(1)}}{\rho^{6}} + ........ \right\} + e^{\alpha_2 \pi\rho} \left\{ \frac{c_1^{(2)}}{\rho^{2}} + \frac{c_2^{(2)}}{\rho^{5}} + \frac{c_3^{(2)}}{\rho^{6}} + ........ \right\} = 0 \quad (2.\ 33)$$

Formula (2. 33) can be written in the form :

$$e^{(\alpha_1 - \alpha_2)\pi\rho} = - \frac{c_1^{(2)} + \frac{c_2^{(2)}}{\rho 3} + \frac{c_3^{(2)}}{\rho 4} + ........}{c_1^{(1)} + \frac{c_2^{(1)}}{\rho 3} + \frac{c_3^{(1)}}{\rho 4} + ........}$$

$$(\alpha_1 - \alpha_2)\,\pi\,\rho = \ln\left(1 + \frac{c_2^{(2)}}{c_1^{(2)}\rho^3} + \frac{c_3^{(2)}}{c_1^{(2)}\rho^4} + ....\right)$$

$$- \ln\left(1 + \frac{c_2^{(1)}}{c_1^{(1)}\rho^3} + \frac{c_3^{(1)}}{c_1^{(1)}\rho^4} + ....\right) + 2k\pi i$$

Upon using ( 2. 31) , we have

$$2\,\pi\,\rho = 2k\pi i + \frac{1}{2\rho^{3}} \int_{o}^{\pi} q\,(t)\,dt + .....$$

$$\rho_k = ik + \frac{1}{\pi 4(ik)^{3}} \int_{o}^{\pi} q\,(t)\,dt + .........$$

$$\rho_k \sim ik\left[1 + \frac{1}{4\pi k^{4}} \int_{0}^{\pi} q(t)dt + .......\right]$$

(3) If $\rho \in S_3 \cup S_4$ , then (2. 30) takes the following form :

$$e^{\alpha_2 \pi\rho}\left\{\frac{c_1^{(2)}}{\rho^2}+\frac{c_2^{(2)}}{\rho^5}+\frac{c_3^{(2)}}{\rho^6}+\ldots\right\}+e^{\alpha_3 \pi\rho}\left\{\frac{c_1^{(3)}}{\rho^2}+\frac{c_2^{(3)}}{\rho^5}+\frac{c_3^{(3)}}{\rho^6}+\ldots\right\}=0 \quad (2.34)$$

Formula (2. 34) can be put in the form

$$e^{(\alpha_3-\alpha_2)\pi\rho}=-\frac{c_1^{(2)}+\frac{c_2^{(2)}}{\rho 3}+\frac{c_3^{(2)}}{\rho 4}+\ldots\ldots}{c_1^{(3)}+\frac{c_2^{(3)}}{\rho 3}+\frac{c_3^{(3)}}{\rho 4}+\ldots\ldots}$$

$$(\alpha_3-\alpha_2)\,\pi\,\rho=2ik\pi\ +\ln\left(1+\frac{c_2^{(2)}}{c_1^{(2)}\rho^3}+\frac{c_3^{(2)}}{c_1^{(2)}\rho^4}+\ldots.\right)$$

$$-\ln\left(1+\frac{c_2^{(3)}}{c_1^{(3)}\rho^3}+\frac{c_3^{(3)}}{c_1^{(3)}\rho^4}+\ldots.\right)$$

From (2. 31) , we have

$$-\ 2i\pi\,\rho=2ik\pi\ -\frac{i}{2\rho^3}\int_0^\pi q\,(t)\,dt+\ldots..$$

then

$$\rho_k\sim-k-\frac{1}{4\pi k^3}\int_0^\pi q\,(t)\,dt+\ldots..$$

$$\rho_k\sim-k\left[1+\frac{1}{4\pi k^4}\int_0^\pi q(t)dt+\ldots\ldots\right]$$

(4) If $\rho \in S_5 \cup S_6$ , then (2. 30) has the following form :

$$e^{\alpha_3 \pi\rho}\left\{\frac{c_1^{(3)}}{\rho^2}+\frac{c_2^{(3)}}{\rho^5}+\frac{c_3^{(3)}}{\rho^6}+\ldots\right\}+e^{\alpha_4 \pi\rho}\left\{\frac{c_1^{(4)}}{\rho^2}+\frac{c_2^{(4)}}{\rho^5}+\frac{c_3^{(4)}}{\rho^6}+\ldots\right\}=0 \quad (2.\ 35)$$

The equation (2. 35) can be written in the form :

$$\overline{e}^{(\alpha_4-\alpha_3)\pi\rho} = -\frac{c_1^{(4)}+\dfrac{c_2^{(4)}}{\rho 3}+\dfrac{c_3^{(4)}}{\rho 4}+\ldots}{c_1^{(3)}+\dfrac{c_2^{(3)}}{\rho 3}+\dfrac{c_3^{(3)}}{\rho 4}+\ldots}$$

$$(\alpha_4 - \alpha_3)\,\pi\,\rho = -\,2ik\pi \ - \ \ln\left(1+\frac{c_2^{(4)}}{c_1^{(4)}\rho^3}+\frac{c_3^{(4)}}{c_1^{(4)}\rho^4}+\ldots.\right)$$

$$+\ln\left(1+\frac{c_2^{(3)}}{c_1^{(3)}\rho^3}+\frac{c_3^{(3)}}{c_1^{(3)}\rho^4}+\ldots\right)$$

Upon using (2. 31) , then

$$2\pi\,\rho = -\,2ik\pi \ +\frac{1}{2\rho^3}\int_0^\pi q\,(t)\,dt+\ldots$$

$$\rho_k \sim -\,ik+\frac{1}{4\pi(-ik)^3}\int_0^\pi q\,(t)\,dt+\ldots..$$

$$\rho_k \sim -\,ik\left[1+\frac{1}{4\pi k^4}\int_0^\pi q(t)dt+\ldots....\right]$$

**2.5. The regularized traces of problem (2. 1 ) - ( 2. 2 ) .**

It is well known that the summation of the diagonal in a square matrix is equal to the summation of eigenvalues of linear operator in finite dimensional space.

In other words , the trace of a matrix is equal to the spectral trace in n - dimensional space .

It is worth mentioning that theorem is satisfied also in the case of nuclear operators which are defined in Hillbert space [30]. Thus we might ask the following question :

Is this theorem applicable in case of unbounded operators ? especially in the case of differential operators the trace of matrices and spectral trace are not exist . For this reason we define the so called " Regularized trace " .
The main role of this section is calculating the regularized traces of eigenvalues of problem (2. 1 ) - ( 2. 2) .

We shall calculate the regularized trace of the operator (2.1) - ( 2. 2 ) , by using the following methods .

**The first method .**

Let us assume that q (x) is an infinitely differentiable function. Then from theorem (2. 2 ) , we have

$$\rho_{k,s} = k\alpha_s \left[1+\frac{1}{4\pi k^4}\int_0^{\pi} q(t)dt+\dots\right] \tag{2. 36}$$

Here the index s ( s = 1 , 2 , 3 , 4 ) denots the index of the sector $T_j$ ( $T_1 = S_0 \cup S_7$ , $T_2 = S_1 \cup S_2$ , $T_3 = S_3 \cup S_4$ , $T_4 = S_5 \cup S_6$ ) in which the given series of roots lies . Raising both sides to the power 4 , we have

$$\rho_{k,s}^4 = k^4\alpha_s^4 \left[1+\frac{1}{4\pi k^4}\int_0^{\pi} q(t)dt+\dots\right]^4$$

$$= k^4\alpha_s^4 \left[1+\frac{1}{\pi k^4}\int_0^{\pi} q(t)dt+\dots\right] \tag{2. 37}$$

Since $\alpha_s^4 = 1$ , then ( 2. 37) can be written in the form

$$\rho_{k,s}^4 = k^4 + \frac{1}{\pi}\int_0^\pi q(t)dt + O(\frac{1}{k^2}) \tag{2. 38}$$

From (2. 38 ) , we deduce that $\sum\limits_{k=1}^{\infty}\sum\limits_{s=1}^{4} \rho_{k,s}^4$ diverges , while

$\sum\limits_{k=1}^{\infty}\left[\sum\limits_{s=1}^{4} \rho_{k,s}^4 - k^4 - \frac{1}{\pi}\int\limits_0^\pi q(t)dt\right]$ converges , then we shall evaluate

$\sum\limits_{k=1}^{\infty}\left[\sum\limits_{s=1}^{4} \rho_{k,s}^4 - k^4 - \frac{1}{\pi}\int\limits_0^\pi q(t)dt\right]$ by using the following theorem

**Theorem2. 3** :The regularized traces of problem (2. 1) -(2.2) are given by the following formula

$$\sum_{k=1}^{\infty}\left[\lambda_k - k^4 - \frac{1}{\pi}\int_0^\pi q(t)dt\right] = -\ \frac{1}{4}(q(o)+q(\pi)) + \frac{1}{2\pi}\int_0^\pi q(t)dt\ .$$

Proof :

Upon using formula (2. 38) , we have

$$\rho_{k,s}^4 = \sum_{n=0}^{\infty} \frac{Q_{2n}^{(s)}}{k^{2n-4}} \tag{2. 39}$$

where

$$Q_0^{(s)} = 1 \ , \ Q_2^{(s)} = 0 \ , \ Q_4^{(s)} = \frac{1}{\pi} \int_o^{\pi} q(t)dt \quad , ....$$

We define the function $\psi_2(-4)$ as follows

$$\psi_2(-4) = \sum_{k=1}^{\infty} \sum_{s=1}^{4} \left[ \rho_{k,s}^4 - \sum_{1=0}^{2} \frac{Q_{21}^{(s)}}{k^{21-4}} \right] \qquad (2.\ 40)$$

Its clear that

$$\psi_2(-4) = Z_1(-4) - Q_2(-4) \ ,$$

where

$$Z_1(-4) = \frac{1}{2\pi i} \oint_{\Gamma_1} \rho^4 \frac{\Delta'(\rho)}{\Delta(\rho)} d\rho \quad , \quad \Gamma_1 \in T_0 \cup T_1$$

$\Gamma_1$ consists of the rayl traversed twice and a circle with center at zero .

And

$$Q_2(-4) = \sum_{k=1}^{\infty} \sum_{s=1}^{4} \sum_{1=0}^{2} \frac{Q_{21}^{(s)}}{k^{21-4}} \ .$$

We note that $Q_2(-4)$ can be expressed in terms of Riemann*'s* zete - function . In fact

$$Q_2(-4) = \sum_{1=0}^{2} Q_{21} \sum_{k=1}^{\infty} \frac{1}{k^{2l-4}} \ .$$

$$Q_2(-4) = \sum_{1=o}^{2} \mathcal{Q}_{21}.\zeta(21-4)$$

$$\mathcal{Q}_{21} = \sum_{s=1}^{4} \mathcal{Q}_{21}^{(s)}$$

Since the values of Zeta - function at the negative integers are well known , then

$$Q_2(-4) = \mathcal{Q}_0.\zeta(-4) + \mathcal{Q}_2.\zeta(-2) + \mathcal{Q}_4.\zeta(0)$$

$$= \left(\frac{4}{\pi}\int_0^{\pi} q(t)dt\right)\left(-\frac{1}{2}\right) = -\frac{2}{\pi}\int_0^{\pi} q(t)\,dt .$$

( see [29] ) .

To calculate $Z_1(-4)$ , we use formula (2. 29) namely

$$\Delta(\rho) = -\frac{1}{\rho^2}(\sin\rho\pi \sinh\rho\pi) + \frac{1}{4\rho^5}(\sinh\rho\pi\cos\rho\pi + \cosh\rho\pi\sin\rho\pi)$$

$$\int_0^{\pi} q(t)\,dt - \frac{1}{4\rho^6}(\sin\rho\pi \ \sinh\rho\pi)(q(0) + q(\pi)) + \ldots \qquad (2.\,41)$$

Differentiate both sides with respect to $\rho$, we have .

$$\Delta'(\rho) = -\frac{\pi}{\rho^2}(\sin\rho\pi \ \cosh\rho\pi + \sinh\rho\pi \ \cos\rho\pi)$$

$$\Bigg[1 - \frac{2\sin\rho\pi\sinh\rho\pi}{\rho\pi(\sin\rho\pi\cosh\rho\pi + \cos\rho\pi\sinh\rho\pi)}$$

$$-\frac{\cos\rho\pi.\cosh\rho\pi}{2\rho^3(\sin\rho\pi\cosh\rho\pi + \cos\rho\pi\sinh\rho\pi)}\int_0^{\pi} q(t)dt + \frac{1}{4\rho^4}(q(0)+q(\pi))$$

$$-\frac{3\sin\rho\pi \ \sinh\rho\pi}{2\pi\rho^5(\sin\rho\pi\cosh\rho\pi + \cos\rho\pi\sinh\rho\pi)}(q(0)+(q(\pi)) \qquad (2.\,42)$$

From (2. 41) and (2. 42) , we obtain

$$\frac{\Delta'(\rho)}{\Delta(\rho)} = \frac{\pi(\sin\rho\pi\,\cosh\rho\pi + \cos\rho\pi\sinh\rho\pi)}{\sin\rho\pi\sinh\rho\pi}$$

$$[1- \frac{2\,\sin\rho\pi\sinh\rho\pi}{\rho\pi(\sin\rho\pi\cosh\rho\pi+\cos\rho\pi\sinh\rho\pi)}$$

$$- \frac{\cos\rho\pi.\cosh\rho\pi}{2\rho^3(\sin\rho\pi\cosh\rho\pi+\cos\rho\pi\sinh\rho\pi)}$$

$$\int_0^{\pi} q(t)dt + \frac{1}{4\rho^4}(q(0)+(q(\pi)) - \frac{3\sin\rho\pi\sinh\rho\pi}{2\pi\rho^5(\sin\rho\pi\cosh\rho\pi+\cos\rho\pi\sinh\rho\pi}$$

$$(q(0)+(q(\pi))+\ldots]$$

$$[1 + \frac{\sin\rho\pi\cosh\rho\pi+\sinh\rho\pi\cos\rho\pi}{4\rho^3\ \sin\rho\pi\sinh\rho\pi} - \frac{1}{4\rho^4}(q(0)+q(\pi))+\ldots]$$

$$\frac{\Delta'(\rho)}{\Delta(\rho)} \sim \sum_{\gamma=0}^{\infty} \frac{w_\gamma}{\rho^\gamma} \tag{2. 43}$$

From formula (2. 43) it is easy to see that

$$w_5 = -(q(0)+q(\pi))) \tag{2. 44}$$

i. e

$$Z_1(-4) = w_5 = -(q(0)+q(\pi))$$

Inserting the values of $Z_1(-4)$, $Q_2(-4)$ in (2. 40) , we have

$$\sum_{k=1}^{\infty}[\sum_{s=1}^{4}\rho_{k,s}^4 - k^4 - \frac{1}{\pi}\int_0^{\pi} q(t)\,dt] = -(q(0)+q(\pi)) + \frac{2}{\pi}\int_0^{\pi} q(t)\,dt.$$

Since $\rho_{k,s}^4 = \lambda_{k,s}$ , and the roots $\rho_{k,s}$ symmetrically situated over all of the sectors$T_s$, then the last formula takes the form

$$\sum_{k=1}^{\infty}[\lambda_k - k^4 - \frac{1}{\pi}\int_0^{\pi} q(t)\,dt] = \frac{1}{2\pi}\int_0^{\pi} q(t)\,dt - \frac{1}{4}(q(0)+(q(\pi)) \tag{2. 45}$$

If $\int_0^{\pi} q(t)\,dt = 0$ , then ( 2. 45) takes the following form

$$\sum_{k=1}^{\infty} [\ \lambda_k - k^4\ ] = -\ \frac{1}{4}\,(\,q(0) + (\,q(\pi)) \qquad (2.\ 46)$$

Formula (2. 46) agree with the result of $\oint$ 17 in [26].

**The second method**

Now we wish to prove formula (2. 45) by using Rouch$e's$ theorem and contour integration .

First ,we shall prove the following Lemmas

**Lemma 2. 2** : If $C_N$ is closed contour which is defined by

$$C_N = \left\{\rho : \left|\operatorname{Re}\rho\right| \le N + \frac{1}{2}, \left|\operatorname{Im}\rho\right| \le N + \frac{1}{2}\right\}.$$

Then $|\cot \rho\pi| < A$ on $C_N$ , where A is constant.

Proof:Suppose that $\rho$=u +i $\upsilon$,we consider the following cases.

1 . If $\upsilon > \frac{1}{2}$, then

$$|\cot \rho\pi| = \left|\frac{e^{i\rho\pi} + e^{-i\rho\pi}}{e^{i\rho\pi} - e^{-i\rho\pi}}\right| = \left|\frac{e^{i\pi u - \pi\upsilon} + \bar{e}^{i\pi u + \pi\upsilon}}{e^{i\pi u - \pi\upsilon} - \bar{e}^{i\pi u + \pi\upsilon}}\right|$$

$$\le \frac{\left|\ e^{i\pi u - \pi\upsilon}\right| + \left|\bar{e}^{i\pi u + \pi\upsilon}\ \right|}{\left|\ e^{-i\pi u + \pi\upsilon}\right| - \left|e^{i\pi u - \pi\upsilon}\ \right|} = \frac{\bar{e}^{\pi\upsilon} + e^{\pi\upsilon}}{e^{\pi\upsilon} - e^{-\pi\upsilon}}$$

$$|\cot \rho\pi| \le \frac{1 + e^{-2\pi\upsilon}}{1 - e^{-2\pi\upsilon}} \le \frac{1 + \bar{e}^{\pi}}{1 - \bar{e}^{\pi}} = A_1$$

2 . If $\upsilon < -\frac{1}{2}$ , then

$$|\cot \rho\pi| \le \frac{|\, e^{i\pi u - \pi v}|+|e^{-i\pi u + \pi v}\,|}{|\, e^{i\pi u - \pi v}|-|e^{-i\pi u + \pi v}\,|} = \frac{\bar{e}^{\pi v} + e^{\pi v}}{\bar{e}^{\pi v} - e^{\pi v}}$$

$$|\cot \rho\pi| \le \frac{1+e^{2\pi v}}{1-e^{2\pi v}} \le \frac{1+\bar{e}^{\pi}}{1+\bar{e}^{\pi}} = A_1$$

3 . If $-\frac{1}{2} \le v \le \frac{1}{2}$, $\rho = N + \frac{1}{2} + iv$ , then we have

$$|\cot \rho\pi| = \left|\cot\pi(N + \frac{1}{2} + iv)\right| = |\cot\,(\,\frac{\pi}{2} + i\,\pi\,\,v\,)\,|$$

$$= |\tanh \pi| \le \tanh \frac{\pi}{2} = A_2$$

4 . If $-\frac{1}{2} \le v \le \frac{1}{2}$ , $\rho = \,-\,N - \frac{1}{2} + iv$ , then we get

$|\cot \rho\pi| \le A_2$ .

If $A = \max\,(\,A_1\,,\,A_2\,)$ , then $|\cot \rho\pi| \;< A$ on $C_N$ , where A is constant which independent on N .

**Lemma 2. 3** : If $|f(\rho)| \le \frac{M}{|\rho^k|}$ on $C_N$, where M , k (k>1) are constants which independent on N. Then

$$\lim_{N\to\infty} \oint_{C_N} \pi\, \cot\rho\,\pi\; f\,(\rho)\, d\rho = 0\,.$$

proof : Upon using lemma (2. 2 ) we get

$$|\,\oint_{C_N} \pi\cot\rho\,\pi f(\rho) d\rho| \le \oint_{C_N} |\,\pi\cot\rho\,\pi|\,|\,f(\rho)|\,|\,d\rho|$$

$$\le \frac{\pi A M}{N^k}(8N+4),\ \text{then}$$

$$\oint_{C_N} \pi\, \cot\rho\,\pi\; f\,(\rho)\, d\rho = 0\,.$$

To prove theorem (2. 3) we can put $\Delta(\rho)$ in the following asymptotic formula

$$\Delta(\rho) = -\,\frac{\sin \rho\pi\,\ \sinh \rho\pi}{\rho^2}\,\left[\,1 + O\,(\,\frac{1}{\rho^3}\,)\,\right] \qquad (2.\,47)$$

since $\rho = 0$ is not an eigenvalue , then for $\sin\rho\pi \ \sinh\rho\pi \neq 0$ the function $\Delta(\rho)$ is an entire function of $\rho$.

Let $D_R$ be the disc in the $\rho$ - Plane with center at the origin and radius $R = N + \frac{1}{2}$ , where N is a positive integer . By Rouchés theorem and asymptotic formula (2. 47) the number of zeros of $\Delta(\rho)$ inside the disc $D_R$ is equal to the number of zeros of the function $\sin \rho\pi \ \sinh\rho\pi$ inside $D_R$.

The charactristic function $\Delta(\rho)$ can be written in the form

$$\Delta(\rho) = - \frac{\sin\rho\pi \ \sinh\rho\pi}{\rho^2} \left[ 1 + \Delta_1(\rho) \right], \tag{2. 48}$$

where

$$\Delta_1(\rho) = - \frac{\sin\rho\pi \ coh\rho\pi + \cos\rho\pi\sinh\rho\pi}{4\rho^3\sin\rho\pi\sinh\rho\pi} \int_0^{\pi} q(t)\, dt$$

$$+ \frac{1}{4\rho^4} \left( q(0) + q(\pi) \right) + \ldots \tag{2. 49}$$

From ( 2. 48) , we have

$$\rho^4 \, d \ \ln \Delta\rho = \rho^4 \left[ d \ \ln(-1) + d \ln \sin\rho\pi + d \ln \ \sinh\rho\pi \right.$$

$$\left. + d \ln \left( 1 + \Delta_1(\rho) \right) - d \ln \rho^2 \right] \tag{2. 50}$$

Now we choose the contour $C_N$ as the following

$$C_N = \left\{ \rho : \left| \mathrm{Re}\,\rho \right| \leq N + \frac{1}{2}, \left| \mathrm{Im}\,\rho \right| \leq N + \frac{1}{2} \right\} \tag{2. 51}$$

See Figure 2

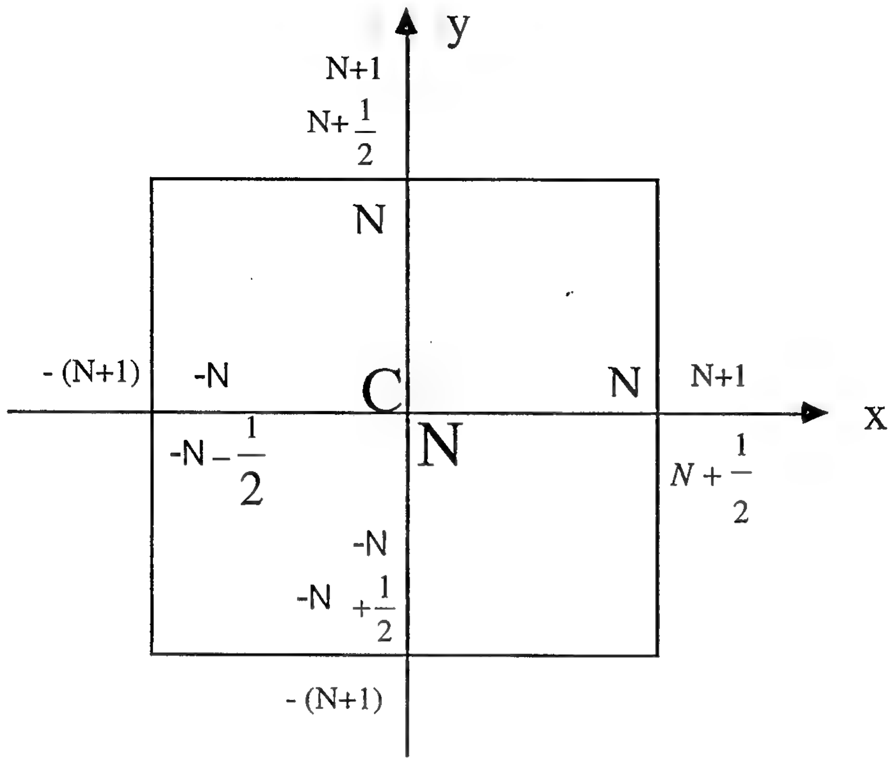



Fig.2

Integrating formula (2. 50) with respect to $\rho$ around the closed contour $C_N$, which is defined by (2. 51) then

$$\frac{1}{2\pi i}\oint_{CN} \rho^4 \frac{\Delta'(\rho)}{\Delta(\rho)} d\rho = \frac{1}{2\pi i}\oint_{CN} \rho^4 \frac{\pi\cos\rho\pi}{\sin\rho\pi} d\rho + \frac{1}{2\pi i}\oint_{CN} \rho^4 \frac{\pi\cosh\rho\pi}{\sinh\rho\pi} d\rho$$

$$+\frac{1}{2\pi i}\oint_{CN} \rho^4 \, d\ln(1+\Delta_1(\rho))\, d\rho . \qquad (2.\ 52)$$

Integrating by parts the integral $\frac{1}{2\pi i}\oint_{CN} \rho^4 \, d \ln(1+\Delta_1(\rho))\, d\rho$, formula (2. 52) can be written in the form

$$\frac{1}{2\pi i}\oint_{CN} \rho^4 \frac{\Delta'(\rho)}{\Delta(\rho)} d\rho = \frac{\pi}{2\pi i}\oint_{CN} \rho^4 \frac{\cos\rho\pi}{\sin\rho\pi} d\rho$$

$$+\frac{\pi}{2\pi i}\oint_{CN} \rho^4 \frac{\cosh\rho\pi}{\sinh\rho\pi} d\rho - \frac{4}{2\pi i}\oint_{CN} \rho^3 \ln(1+\Delta_1(\rho))\, d\rho . \ (2.\ 53)$$

Suppose that , the roots of $\Delta(\rho)$ are simple , then using the residue theorem ( Theorem 1. 3 in chapter I ) , we have

$$\sum_{k=1}^{N} \rho_k^4 = 2 \sum_{k=1}^{N} k^4 + \lim_{\rho\to 0} \frac{\pi\rho^5\cos\rho\pi}{\sin\rho\pi} + 2 \sum_{k=1}^{N} (ik)^4$$

$$+\lim_{\rho\to 0}\frac{\rho^5\pi\cosh\rho\pi}{\sinh\rho\pi} - \frac{4}{2\pi i}\oint_{CN} \rho^3 \ln(1+\Delta_1(\rho))\, d\rho. \quad (2.54)$$

By using *L'hospital's* rule , we have

$$4\sum_{k=1}^{N} \lambda_k = 4\sum_{k=1}^{N} k^4 - \frac{4}{2\pi i}\oint_{CN} \rho^3[\Delta_1(\rho) - \frac{\Delta_1^2(\rho)}{2} + \frac{\Delta_1^3(\rho)}{3} - \ldots]d\rho \quad (2.55)$$

Upon using formula (2. 49) , we get

$$\Delta_1(\rho) - \frac{\Delta_1^2(\rho)}{2} + \frac{\Delta_1^3(\rho)}{3} - \ldots =$$

$$-\frac{(\sin\rho\pi\ \coh\rho\pi + \cos\rho\pi\sinh\rho\pi)}{4\rho^3\sin\rho\pi\sinh\rho\pi}\int_0^\pi q(t)\,dt + \frac{1}{4\rho^4}(q(0)+q(\pi)) + \ldots$$

$$-\frac{1}{2}\Big[\frac{(\sin\rho\pi\ coh\rho\pi + \cos\rho\pi\sinh\rho\pi)^2}{16\rho^6\sin^2\rho\pi\sinh^2\rho\pi}\Big(\int_0^\pi q(t)\,dt\Big)^2$$

$$+\frac{1}{16\rho^8}(q(0)+q(\pi))^2 + \ldots\Big] + \ldots$$

Then (2. 55) takes the following form

$$4\sum_{k=1}^{N} \lambda_k = 4\sum_{k=1}^{N} k^4 - \frac{4}{2\pi i}\oint_{CN}\Big[-\frac{1}{4\sin\rho\pi\sinh\rho\pi}$$

$$(\sin\rho\pi\cosh\rho\pi + \cos\rho\pi\sinh\rho\pi)\int_0^\pi q(t)\,dt$$

$$+\frac{1}{4\rho}(q(0)+q(\pi)) + \ldots\Big]d\rho$$

$$= 4\sum_{k=1}^{N} k^4 + \frac{\int_0^\pi q(t)dt}{2\pi i}\oint_{CN}[\coth\rho\pi + \cot\rho\pi]\,d\rho$$

$$-(q(0)+q(\pi)) + f(N), \quad (2.56)$$

Upon using the residue theorem , we have

$$\frac{1}{2\pi i}\oint_{C_N}[\cot\rho\pi+\coth\rho\pi]d\rho=2\sum_{k=1}^{N}\lim_{\rho\to k}\frac{(\rho-k)\cos\rho\pi}{\sin\rho\pi}+$$

$$+\lim_{\rho\to 0}\frac{\rho\cos\rho\pi}{\sin\rho\pi}$$

$$+2\sum_{k=1}^{N}\lim_{\rho\to ik}(\rho-ik)\frac{\cosh\rho\pi}{\sinh\rho\pi}$$

$$+\lim_{\rho\to 0}\rho\frac{\cosh\rho\pi}{\sinh\rho\pi}=\frac{4N}{\pi}+2. \qquad (2.57)$$

Inserting (2. 57) in (2. 56) , we get

$$\lim_{N\to\infty}\sum_{k=1}^{N}\left[\lambda_k-k^4-\frac{1}{\pi}\int_o^\pi q(t)dt\right]=-\frac{1}{4}(q(0)+q(\pi))+\frac{1}{2\pi}\int_o^\pi q(t)dt$$

$$+\lim_{N\to\infty}f(N)$$

Upon using Lemmas (2.2) and (2.3),we can prove that

$\lim_{N\to\infty} f(N)=0$ .

Then the regularized traces for the problem (2.1) - (2.2) are given by the following formula .

$$\sum_{k=1}^{\infty}\left[\lambda_k-k^4-\frac{1}{\pi}\int_o^\pi q(t)dt\right]=\frac{1}{2\pi}\int_o^\pi q(t)dt-\frac{1}{4}(q(0)+q(\pi)).$$

**2.6.The regularized sum of the eigenfunction of problem (2. 1)- (2. 2) .**

There is an interesting connection between the eigenfunction and the Gree$n'$s function .

If all the eigenfunctions are known , we can find the Gree$n'$s function .

The converse is also true in the sense that knowledge of the Green's function G $(x, \xi, \rho)$ for all complex values of the parameter $\rho$ leads to the eigenfunction and consequently to the eigenfunction expansion. That expansion is obtained by integrating G $(x, \xi, \rho)$ around a large contour in the complex $\rho$ - plane .

In the following we obtained the Green's function of the problem (2. 1 ) - (2. 2) and by using it we can find the regularized sum for the eigenfunction

**lemma 2. 4** : The Green's function of problem(2. 1) - (2. 2) is given by the following formula

$$G(x, \xi, \rho) = g(x,\xi) - g(0,\xi)\ \phi_1(x,\rho) - g''(0,\xi)\ \phi_3(x,\rho) + \frac{1}{\Delta(\rho)}\{\ g(0,\xi)[\phi_2(x,\rho)(\phi''_1(\pi,\rho)\phi_4(\pi,\rho) - \phi''_4(\pi,\rho)\ \phi_1(\pi,\rho)) + \phi_4(x,\rho)(\phi''_2(\pi,\rho)\phi_1(\pi,\rho) - \phi''_1(\pi,\rho)\phi_2(\pi,\rho))] + g''(o,\xi)[\ \phi_2(\pi,\rho)\ (\phi''_3(\pi,\rho)\ \phi_4(\pi,\rho) - \phi''_4(\pi,\rho)\phi_3(\pi,\rho)) + \phi_4(x,\rho)(\phi''_2(\pi,\rho)\phi_3(\pi,\rho) - \phi''_3(\pi,\rho)\phi_2(\pi,\rho))] + g(\pi,\xi)[\ \phi_2(x,\rho)\ \phi''_4(\pi,\rho) - \phi_4(x,\rho)\ \phi''_2(\pi,\rho)] + g''(\pi,\xi)[\phi_4(x,\rho)\ \phi_2(\pi,\rho) - \phi_2(x,\rho)\ \phi_4(x,\rho)]\ \}. \tag{2. 58}$$

where g $(x, \xi)$ is given by formula (1. 21) , and the functions $\phi_j^{(\gamma-1)}(x,\rho)$ ( $j = 1, 2, 3, 4$, $\gamma = 1, 2, 3, 4$ ) are given by the formulae(2. 9), (2. 18 ),( 2. 20) and (2. 21) .

Proof : Upou using formula (1. 18) , with n = 4 , we have

$$G( x , \xi , \rho) = \frac{H(x,\xi,\rho)}{\Delta(\rho)} , \qquad (2.59)$$

where

$$\Delta(\rho) = \phi_4 ( \pi ,\rho ) \phi''_2(\pi,\rho) - \phi_2 ( \pi ,\rho) \phi''_4 (\pi,\rho) ,$$

and

$$H(x,\xi,\rho) = \begin{vmatrix} \phi_1(x,\rho) & \phi_2(x,\rho) & \phi_3(x,\rho) & \phi_4(x,\rho) & g(x,\xi) \\ 1 & 0 & 0 & 0 & g(0,\xi) \\ 0 & 0 & 1 & 0 & g''(0,\xi) \\ \phi_1(\pi,\rho) & \phi_2(\pi,\rho) & \phi_3(\pi,\rho) & \phi_4(\pi,\rho) & g(\pi,\xi) \\ \phi''_1(\pi,\rho) & \phi''_2(\pi,\rho) & \phi''_3(\pi,\rho) & \phi''_4(\pi,\rho) & g''(\pi,\xi) \end{vmatrix}$$

$$= - \begin{vmatrix} \phi_2(x,\rho) & \phi_3(x,\rho) & \phi_4(x,\rho) & g(x,\xi) \\ 0 & 1 & 0 & g''(0,\xi) \\ \phi_2(\pi,\rho) & \phi_3(\pi,\rho) & \phi_4(\pi,\rho) & g(\pi,\xi) \\ \phi''_2(\pi,\rho) & \phi''_3(\pi,\rho) & \phi''_4(\pi,\rho) & g''(\pi,\xi) \end{vmatrix}$$

$$- g(0,\xi) \begin{vmatrix} \phi_1(x,\rho) & \phi_2(x,\rho) & \phi_3(x,\rho) & \phi_4(x,\rho) \\ 0 & 0 & 1 & 0 \\ \phi_1(\pi,\rho) & \phi_2(\pi,\rho) & \phi_3(\pi,\rho) & \phi_4(\pi,\rho) \\ \phi''_1(\pi,\rho) & \phi''_2(\pi,\rho) & \phi''_3(\pi,\rho) & \phi''_4(\pi,\rho) \end{vmatrix}$$

Hence

$$H(x,\xi,\rho)=-\begin{vmatrix}\phi_2(x,\rho) & \phi_4(x,\rho) & g(x,\xi)\\ \phi_2(\pi,\rho) & \phi_4(\pi,\rho) & g(\pi,\xi)\\ \phi''_2(\pi,\rho) & \phi''_4(\pi,\rho) & g''(\pi,\xi)\end{vmatrix}$$

$$-g''(0,\xi)\begin{vmatrix}\phi_2(x,\rho) & \phi_3(x,\rho) & \phi_4(x,\rho)\\ \phi_2(\pi,\rho) & \phi_3(\pi,\rho) & \phi_4(\pi,\rho)\\ \phi''_2(\pi,\rho) & \phi''_3(\pi,\rho) & \phi''_4(\pi,\rho)\end{vmatrix}$$

$$+g(0,\xi)\begin{vmatrix}\phi_1(x,\rho) & \phi_2(x,\rho) & \phi_4(x,\rho)\\ \phi_1(\pi,\rho) & \phi_2(\pi,\rho) & \phi_4(\pi,\rho)\\ \phi''_1(\pi,\rho) & \phi''_2(\pi,\rho) & \phi''_4(\pi,\rho)\end{vmatrix}$$

$$\begin{aligned}H(x,\xi,\rho) &= g(x,\xi)\,\Delta(\rho) - g(0,\xi)\,\phi_1(x,\rho)\,\Delta(\rho)\\ &\quad - g''(0,\xi)\,\phi_3(x,\rho)\Delta(\rho)\\ &\quad + g(0,\xi)[\phi_2(x,\rho)(\phi''_1(\pi,\rho)\phi_4(\pi,\rho) - \phi''_4(\pi,\rho)\phi_1(\pi,\rho))\\ &\quad + \phi_4(x,\rho)(\phi_1(\pi,\rho)\,\phi''_2(\pi,\rho)\\ &\quad - \phi_2(\pi,\rho)\,\phi''_1(\pi,\rho))]\\ &\quad + g''(0,\xi)[(\phi_2(x,\rho)(\phi_4(\pi,\rho)\,\phi''_3(\pi,\rho)\\ &\quad - \phi_3(\pi,\rho)\,\phi''_4(\pi,\rho))\end{aligned}$$

$$+\phi_4(x,\rho)(\phi_3(\pi,\rho)\phi''_2(\pi,\rho)$$

$$-\phi_2(\pi,\rho)\phi''_3(\pi,\rho))]$$

$$+g(\pi,\xi)[\phi_2(x,\rho)\phi''_4(\pi,\rho)$$

$$-\phi_4(x,\rho)\phi''_2(\pi,\rho)]$$

$$+g''(\pi,\xi)[\phi_4(x,\rho)\phi_2(\pi,\rho)$$

$$-\phi_2(x,\rho)\phi_4(\pi,\rho)]$$

Inserting the functions H $(x,\xi,\rho)$ and $\Delta(\rho)$ in formula (2. 59) , we get the proof of lemma ( 2. 4 ) .

**Lemma 2.5** : For the function $g(x,\xi)$ and its derivatives with respect to x at the points $x=0,\pi$ has the following asymptotic formulae

$$g(x,\xi)=\pm\frac{1}{2}\Big[\frac{1}{2\rho^3}(\sin\rho(\xi-x)+\sinh\rho(x-\xi))$$
$$+\frac{1}{8\rho^6}(\cosh\rho(x-\xi)-\cos\rho(x-\xi))\Big(\int_0^{\xi}q(t)dt-\int_0^{x}q(t)dt\Big)$$
$$+\frac{3}{16\rho^7}(q(x)+q(\xi))(\sinh\rho(x-\xi)-\sin\rho(x-\xi))+\ldots\Big], \quad (2.\ 60)$$

where the positive sign being taken if $x>\xi$, and the negative sign if $x<\xi$ . For example

$$g(0,\xi)=-\frac{1}{2}\Big[\frac{1}{2\rho^3}(\sin\rho\xi-\sinh\rho\xi)+\frac{1}{8\rho^6}(\cosh\rho\xi$$
$$-\cos\rho\xi)\int_0^{\xi}q(t)dt+\frac{3}{16\rho^7}(q(0)+q(\xi))(\sin\rho\xi-\sinh\rho\xi)+\ldots\Big], \quad (2.\ 61)$$

$$g(\pi,\xi)=\frac{1}{2}\Big[\frac{1}{2\rho^3}(\sinh\rho(\pi-\xi)-\sin\rho(\pi-\xi))$$
$$-\frac{1}{8\rho^6}(\cosh\rho(\pi-\xi)-\cos\rho(\pi-\xi))\int_\xi^\pi q(t)dt$$
$$+\frac{3}{16\rho^7}(q(\pi)+q(\xi))(\sinh\rho(\pi-\xi)-\sin\rho(\pi-\xi))+\ldots\Big],$$

(2. 62)

But for the functions $g''(0,\xi)$ and $g''(\pi,\xi)$, we have

$$g''(0,\xi)=\frac{1}{2}\Big[\frac{1}{2\rho}(\sin\rho\xi+\sinh\rho\xi)-\frac{(\cos\rho\xi+\cosh\rho\xi)}{8\rho^4}$$
$$\int_0^\xi q(t)dt+\frac{1}{16\rho^5}(3q(\xi)-q(0))(\sin\rho\xi+\sinh\rho\xi)+\frac{q'(0)}{4\rho^6}$$
$$(\cos\rho\xi-\cosh\rho\xi)+\frac{3}{16\rho^7}q''(0)(\sinh\rho\xi-\sin\rho\xi)+\ldots\Big] \quad (2.\ 63)$$

Proof : Using formula (1. 21) , we have

$$g(x,\xi)=\pm\frac{2}{2W(\xi)}\begin{vmatrix}\phi_1(x,\rho) & \phi_2(x,\rho) & \phi_3(x,\rho) & \phi_4(x,\rho)\\ \phi''_1(\xi,\rho) & \phi''_2(\xi,\rho) & \phi''_3(\xi,\rho) & \phi''_4(\xi,\rho)\\ \phi'_1(\xi,\rho) & \phi'_2(\xi,\rho) & \phi'_3(\xi,\rho) & \phi'_4(\xi,\rho)\\ \phi_1(\xi,\rho) & \phi_2(\xi,\rho) & \phi_3(\xi,\rho) & \phi_4(\xi,\rho)\end{vmatrix}$$

$$W(\xi)=\begin{vmatrix}\phi'''_1(\xi,\rho) & \phi'''_2(\xi,\rho) & \phi'''_3(\xi,\rho) & \phi'''_4(\xi,\rho)\\ \phi''_1(\xi,\rho) & \phi''_2(\xi,\rho) & \phi''_3(\xi,\rho) & \phi''_4(\xi,\rho)\\ \phi'_1(\xi,\rho) & \phi'_2(\xi,\rho) & \phi'_3(\xi,\rho) & \phi'_4(\xi,\rho)\\ \phi_1(\xi,\rho) & \phi_2(\xi,\rho) & \phi_3(\xi,\rho) & \phi_4(\xi,\rho)\end{vmatrix}=W(\phi_1,\phi_2,\phi_3,\phi_4)$$

where W $(\phi_1,\phi_2,\phi_3,\phi_4)$ is the Wronskian determinant of the solutions $\phi_1$, $\phi_2$, $\phi_3$ and $\phi_4$ . Since the coefficient of $y'''$ in the differential equation ( 2. 1 ) is equal to zero , then W($\xi$) does not depend on $\xi$ . Therefore W($\xi$) = W(0) :
W(0) =W($\phi_1(0,\rho),(\phi_2(0,\rho),(\phi_3(0,\rho),(\phi_4(0,\rho)) = 1$ .

Then

$$g(x,\xi)=\pm\frac{1}{2}\begin{vmatrix} \phi_1(x,\rho) & \phi_2(x,\rho) & \phi_3(x,\rho) & \phi_4(x,\rho) \\ \phi''_1(\xi,\rho) & \phi''_2(\xi,\rho) & \phi''_3(\xi,\rho) & \phi''_4(\xi,\rho) \\ \phi'_1(\xi,\rho) & \phi'_2(\xi,\rho) & \phi'_3(\xi,\rho) & \phi'_4(\xi,\rho) \\ \phi_1(\xi,\rho) & \phi_2(\xi,\rho) & \phi_3(\xi,\rho) & \phi_4(\xi,\rho) \end{vmatrix}$$

Upon expanding this determinant and using the formulae (2. 9) , (2. 18) and (2. 20) , we get the formulae (2. 60) , (2. 61) and (2. 62) , the Lemma ( 2. 5) is proved .
Suppose that $\rho_{k,s}$ (s = 1,2,3,4) are simple zeros of the function $\Delta(\rho)$ . Then $\rho_{k,s}$ are simple poles for the Gree$n's$ function G(x,$\xi$, $\rho$) such that

$$G(x,\xi,\rho)=\frac{R(x,\xi,\rho)}{\rho-\rho_{k,s}}+G_1(x,\xi,\rho),$$

where $G_1(x,\xi,\rho)$ is regular function in the neighborhood of the points $\rho_{k,s}$ . Upon using the residue theorem (1.3) in chapter I , we have

$$R(x, \xi, \rho_{k,s}) = \frac{H(x,\xi,\rho_{k,s})}{\Delta'(\rho_{k,s})} .$$

Since

$$H(x,\xi,\rho_{k,s}) = -\begin{vmatrix} \phi_2(x,\rho_{k,s}) & \phi_3(x,\rho_{k,s}) & \phi_4(x,\rho_{k,s}) & g(x,\xi) \\ 0 & 1 & 0 & g''(0,\xi) \\ \phi_2(\pi,\rho k,s) & \phi_3(\pi\rho,k,s) & \phi_4(\pi,\rho k,s) & g(\pi,\xi) \\ \phi''_2(\pi,\rho k,s) & \phi''_3(\pi,\rho k,s) & \phi''_4(\pi,\rho k,s) & g''(\pi,\xi) \end{vmatrix}$$

Then for fixed $\xi$,we can prove that the function $R(x, \xi,\rho_{k,s})$ satisfies the differential equation $1(R) = \rho^4 R$,
and the boundary conditions

$$R(0, \xi, \rho_{k,s}) = 0 , \ R''(0, \xi, \rho_{k,s}) = 0 ,$$
$$R(\pi, \xi, \rho_{k,s}) = 0 , \ R''(\pi, \xi, \rho_{k,s}) = 0 ,$$

we deduce that the functions $R(x, \xi, \rho_{k,s})$ are eigenfunctions corresponding to the eigenvalues $\rho_{k,s}$

Upon using Lemma (2. 4) and Lemma (2. 5) , we have

$$R(x, \xi,\rho_{k,s}) = \frac{1}{\Delta'(\rho_{k,s})} \left[ \frac{M_5(\rho_{k,s})}{\rho^5_{k,s}} + \frac{M_8(\rho_{k,s})}{\rho^8_{k,s}} + \dots \right] , \qquad (2.64)$$

where

$M_5(\rho) = \frac{1}{4} [ \sin\rho\xi \sin\rho x \sinh\rho\pi \cos\rho\pi$

$- \sinh \rho \xi \sinh \rho x \cosh \rho \pi \ \sin \rho \pi$

$+ \sinh \rho ( \pi - \xi ) \sin \rho \pi \ \sinh \rho x$

$- \ \sin \rho ( \pi - \xi ) \sin \rho x \ \sinh \rho \pi \ ]$ ,

$M_8(\rho) = \frac{1}{16} \int_0^\pi q(t)\, dt [ \sin \rho \xi \sin \rho x ( \sin \rho \pi \sinh \rho \pi$

$- \cos \rho \pi \cosh \rho \pi ) + \sinh \rho \xi \sinh \rho x ( \sin \rho \pi \sinh \rho \pi$

$+ \cos \rho \pi \ \cosh \rho \pi ) + \cos \rho ( \pi - \xi ) \sin \rho x \sinh \rho \pi$

$- \cosh \rho ( \pi - \xi ) \sinh \rho x \sin \rho \pi ]$ ,

and $\Delta'(\rho)$ is given by formula (2. 42)

**Theorem 2.4** :The sum of eigenfunctions of problem (2. 1)- (2. 2) are given by the following formula

$$\sum_{k=1}^{\infty} \sum_{s=1}^{4} [ \lambda_{k,s} R(x , \xi, \rho_{k,s}) ] = 0 .$$

proof :

Let $\rho = \rho_{k,1} = k [ 1 + \delta_k ]$, $\delta_k = \frac{1}{4\pi k^4} \int_0^\pi q \, (t)\, dt$,

then

$\cos \rho \pi = \cos \pi ( k + \delta_k k) \sim ( - )^k \cos k\pi \ \delta_k$ ,

$\sin \rho \pi = \sin \pi ( k + \delta_k k) \sim ( - )^k \sin k \pi \ \delta_k$ ,

$\sinh \rho \pi = \sinh \pi ( k + \delta_k k) \sim \frac{1}{2} e^{k\pi}$

and

$\cosh \rho \pi = \cosh \pi ( k + \delta_k k) \sim \frac{1}{2} e^{k\pi}$

From (2. 42) , we have

$$\Delta'(\rho) = -\frac{\pi}{\rho^2} \cos \rho \pi \ \sinh \rho \pi$$

$$\left[ 1 + \frac{\sin \rho\pi \cosh \rho\pi}{\cos \rho\pi \sinh \rho\pi} - \frac{2 \sin \rho\pi}{\pi\rho \cos \rho\pi} - \frac{\cosh \rho\pi}{2\rho^3 \sinh \rho\pi} \int_0^\pi q(t)dt + \ldots \right]$$

$$\left[\Delta'(\rho)\right]^{-1} = \frac{-\rho^2}{\pi\cos\rho\pi\sinh\rho\pi}\left[1-\tan\rho\pi\coth\rho\pi+\frac{2\sin\rho\pi}{\rho\pi\cos\rho\pi}+\dots\right]$$

$$\left[\Delta'(\rho_{k,1})\right]^{-1} \sim \frac{-2\rho_{k,1}^2}{\pi e^{k\pi}(-1)^k(\cos k\pi\delta_k)}\left[1-\tan k\pi\delta_k . 1+2\frac{\tan k\pi\,\delta_k}{\pi\rho_{k,1}}+\dots\right] \tag{2. 65}$$

$$\rho_{k,1}^4 R(\text{x},\xi,\rho_{k,1}) =$$

$$\frac{-2\rho_{k,1}^6}{\pi(-)^k e^{k\pi}\cos k\pi\delta_k}\left[\frac{M_5(\text{x}\xi,\rho_{k,1})}{\rho_{k,1}^5}+\frac{M_8(\text{x}\xi,\rho_{k,1})}{\rho_{k,1}^8}+\dots\right]$$

$$\left[1-\tan k\pi\delta_k+\frac{2}{\pi\rho_{k,1}}\tan k\pi\delta_k+\frac{1}{2\rho_{k,1}^3}\int_0^\pi q(t)dt+\dots\right] \tag{2. 66}$$

where

$$M_5(\text{x},\xi,\rho_{k,1}) \sim \frac{2}{4}\ \sin\rho_{k,1}\text{x}\ \sin\rho_{k,1}\ (-1)^k\left(\frac{1}{2}\right) e^{k\pi}\ \cos\rho k\pi\delta_k$$

$$\rho_{k,1}^4 R(\text{x},\xi,\rho_{k,1}) \sim -\frac{1}{2\pi}\ \rho_{k,1}\sin\rho_{k,1}\text{x}\sin\rho_{k,1}\xi\ (\ 1-\tan k\pi\,\delta_k),$$

we note that $\rho_{k,3} = -\ \rho_{k,1}$, then

$$\rho_{k,3}^4 R(\text{x},\xi,\rho_{k,3}) \sim \frac{1}{2\pi}\ \rho_{k,1}\sin\rho_{k,1}\text{x}\sin\rho_{k,1}\xi\ \ (\ 1-\tan k\pi\,\delta_k),$$

by the same way , we have

$$\rho_{k,2}^4 R(\text{x},\xi,\rho_{k,2}) = -\ \rho_{k,4}^4 R(\text{x},\xi,\rho_{k,4}),\ \text{then}$$

$$\sum_{k=1}^{\infty}\ \sum_{s=1}^{4}\left[\ \rho_{k,s}^4 R(\text{x},\xi,\rho_{k,s})\ \right] = 0\ .$$

# CHAPTERIII

## REGULARIZED TRACES AND SUMS OF EIGENFUNCTIONS FOR PERIODIC BOUNDARY CONDITIONS

### 3.1. Introduction .

In this chapter we will investigate the asymptotic behaviour of the eigenvalues of $l(y) = \lambda\ y$ for large $|\lambda|$ , where l(y) is a linear differential expression in the form

$$l(y) = y^{(4)}(x) + q(x)\, y\ ,\ 0 \le x \le \pi \tag{3. 1}$$

and the boundary conditions

$$\begin{aligned} U_1(y) &= y(\pi) - y(0) = 0\ , \\ U_2(y) &= y'(\pi) - y'(0) = 0\ , \\ U_3(y) &= y''(\pi) - y''(0) = 0\ , \\ U_4(y) &= y'''(\pi) - y'''(0) = 0\ , \end{aligned} \tag{3. 2}$$

Also we shall calculate the regularized sums of eigenfunctions of the problem (3. 1) - (3. 2) .

First , we put $\lambda = \rho^4$ ; the equation $l(y) = \lambda\ y$ takes the form

$$y^{(4)}(x) + q(x)\, y = \rho^4\ y \tag{3. 3}$$

We note that theorem(2. 1) and lemma (2. 1) are holds.

**3.2 . Asymptotic behaviour of the eigenvalues of problem (3. 1) - (3. 2) .**

**Theorem 3. 1** : Adifferential operator of the fourth order which has precisely denumerably many eigenvalues , whose behaviour at infinity is specified by the following formulae

(1) If $\rho \in T_0$ , then

$$\rho_{k,0} \sim (-1+i)(k+\frac{1}{4})\left[1-\frac{1}{16\pi(k+\frac{1}{4})^4}\int_0^\pi q(t)dt+O(\frac{1}{k^6})\right] \quad (3.4)$$

(2) If $\rho \in T_1$ , then

$$\rho_{k,1} \sim i(k+\frac{1}{4})\left[1+\frac{1}{4\pi(k+\frac{1}{4})^4}\int_0^\pi q(t)dt+O(\frac{1}{k^6})\right] \quad (3.5)$$

(3) If $\rho \in T_2$ , then

$$\rho_{k,2} \sim (1+i)(k+\frac{1}{4})\left[1-\frac{1}{16\pi(k+\frac{1}{4})^4}\int_0^\pi q(t)dt+O(\frac{1}{k^6})\right] \quad (3.6)$$

(4) If $\rho \in T_3$ , then

$$\rho_{k,3} \sim (k+\frac{1}{4})\left[1+\frac{1}{4\pi(k+\frac{1}{4})^4}\int_0^\pi q(t)dt+O(\frac{1}{k^6})\right] \quad (3.7)$$

(5) If $\rho \in T_4$ , then

$$\rho_{k,4} \sim (1-i)(k+\frac{1}{4})\left[1-\frac{1}{16\pi(k+\frac{1}{4})^4}\int_0^\pi q(t)dt+O(\frac{1}{k^6})\right] \quad (3.8)$$

(6) If $\rho \in T_5$ , then

$$\rho_{k,5} \sim -i(k+\frac{1}{4})\left[1+\frac{1}{4\pi(k+\frac{1}{4})^4}\int_0^\pi q(t)dt+O(\frac{1}{k^6})\right] \quad (3.9)$$

(7) If $\rho \in T_6$, then

$$\rho_{k,6} \sim -(1+i)(k+\frac{1}{4})\left[1-\frac{1}{16\pi(k+\frac{1}{4})^4}\int_0^{\pi} q(t)dt+O(\frac{1}{k^6})\right] \tag{3.10}$$

(8) If $\rho \in T_7$, then

$$\rho_{k,7} \sim -(k+\frac{1}{4})\left[1+\frac{1}{4\pi(k+\frac{1}{4})^4}\int_0^{\pi} q(t)dt+O(\frac{1}{k^6})\right] \tag{3. 11}$$

Proof : Upon using formula (1. 12) in chaptr I, we have

$$\Delta(\rho) = \begin{vmatrix} U_1(\phi_1) & U_1(\phi_2) & U_1(\phi_3) & U_1(\phi_4) \\ U_2(\phi_1) & U_2(\phi_2) & U_2(\phi_3) & U_2(\phi_4) \\ U_3(\phi_1) & U_3(\phi_2) & U_3(\phi_3) & U_3(\phi_4) \\ U_4(\phi_1) & U_4(\phi_2) & U_4(\phi_3) & U_4(\phi_4) \end{vmatrix} \tag{3. 12}$$

But from (3. 2) and the conditions (2. 8) , we get

$$U_1(\phi_1)=\phi_1(\pi)-1, U_2(\phi_1)=\phi'_1(\pi), U_3(\phi_1)=\phi''_1(\pi), U_4(\phi_1)=\phi'''_1(\pi),$$
$$U_1(\phi_2)=\phi_2(\pi), U_2(\phi_2)=\phi'_2(\pi)-1, U_3(\phi_2)=\phi''_2(\pi), U_4(\phi_2)=\phi'''_2(\pi),$$
$$U_1(\phi_3)=\phi_3(\pi), U_2(\phi_3)=\phi'_3(\pi), U_3(\phi_3)=\phi''_3(\pi)-1, U_4(\phi_3)=\phi'''_3(\pi),$$
$$U_1(\phi_4)=\phi_4(\pi), U_2(\phi_4)=\phi'_4(\pi), U_3(\phi_4)=\phi''_4(\pi), U_4(\phi_4)=\phi'''_4(\pi)-1. \tag{3. 13}$$

Substituting (3. 13) in (3. 12) , we have the following formula for $\Delta(\rho)$

$$\Delta(\rho)=\begin{vmatrix} \phi_1(\pi)-1 & \phi_2(\pi) & \phi_3(\pi) & \phi_4(\pi) \\ \phi'_1(\pi) & \phi'_2(\pi)-1 & \phi'_3(\pi) & \phi'_4(\pi) \\ \phi''_1(\pi) & \phi''_2(\pi) & \phi''_3(\pi)-1 & \phi''_4(\pi) \\ \phi'''_1(\pi) & \phi'''_2(\pi) & \phi'''_3(\pi) & \phi'''_4(\pi)-1 \end{vmatrix} \quad (3.14)$$

Now inserting the values of the functions $\phi_j^{(k)}(\pi)$, (k = 0, 1 , 2 , 3 ; j = 1, 2 , 3, 4) from formulae (2. 9) , ( 2. 18) , (2. 20) and (2. 21) into (3. 14) , we obtain the following equation for the determination of the eigenvalues is given by the following formula .

$$\begin{aligned}\Delta(\rho) &= 4+4\cosh\rho\pi\cos\rho\pi - \frac{7}{2}(\cos\rho\pi+\cosh\rho\pi) \\ &- \frac{\cos\rho\pi}{2}(\sin\rho\pi\sinh\rho\pi+\cos\rho\pi\cosh\rho\pi)+ \\ &\frac{\cosh\rho\pi}{2}(\sin\rho\pi\sinh\rho\pi-\cos\rho\pi\cosh\rho\pi)+ \\ &\frac{\int_0^\pi q(t)dt}{16\rho^3}\Big[12(\sinh\rho\pi-\sin\rho\pi)+ \\ &16(\cosh\rho\pi\sin\rho\pi-\sinh\rho\pi\cos\rho\pi)- \\ &\cosh\rho\pi\sin\rho\pi(6\cosh\rho\pi+2\cos\rho\pi)+ \\ &\sinh\rho\pi\cos\rho\pi(6\cos\rho\pi+2\cosh\rho\pi)\Big] \\ &+\frac{(q(o)-q(\pi))}{32\rho^4}\Big[6\sin\rho\pi\cosh\rho\pi(\sin\rho\pi+\sinh\rho\pi) \\ &-6\sinh\rho\pi\cos\rho\pi(\sin\rho\pi+\sinh\rho\pi)\Big]+O\left(\frac{1}{\rho^6}\right)=0 \quad (3.15)\end{aligned}$$

The equation (3. 15) can be written in the form

$$\Delta(\rho) = \sum_{k=1}^{12} e^{\alpha_k \rho\pi} \left\{ c_1^{(k)} + \frac{c_2^{(k)}}{\rho^3} + \frac{c_3^{(k)}}{\rho^4} + \ldots \right\}. \tag{3. 16}$$

Let $\Gamma$ be the complex hull of the points $\overline{\alpha}_1^{(0)}$, $\overline{\alpha}_2^{(0)}$, $\alpha_3^{(0)}$, $\alpha_4^{(0)}$, $\alpha_5^{(0)}$, $\overline{\alpha}_6^{(0)}$, $\overline{\alpha}_7^{(0)}$, and $\overline{\alpha}_8^{(0)}$ . It is easy to see that the zeros of $\Delta(\rho)$ asymptotically coinside with the zeros of the function F ( $\rho$) which is given by the following formula

$$F(\rho) = \sum_{k=1}^{8} e^{\overline{\alpha}_k^{(0)} \rho\pi} \left\{ \overline{c}_1^{(k)} + \frac{\overline{c}_2^{(k)}}{\rho^3} + \frac{\overline{c}_3^{(k)}}{\rho^4} + \ldots \right\}, \tag{3. 17}$$

where

$\overline{\alpha}_1^{(0)} = 2 + i$ , $\overline{\alpha}_2^{(0)} = 1 + 2i$ , $\overline{\alpha}_3^{(0)} = -1 + 2i$ , $\overline{\alpha}_4^{(0)} = -2+i$,

$\overline{\alpha}_5^{(0)}, = -2-i$, $\overline{\alpha}_6^{(0)} = -1 -2i$, $\overline{\alpha}_7^{(0)} = 1- 2i$ , $\overline{\alpha}_8^{(0)} = 2-i$ ,

$$\overline{c}_1^{(1)} = \overline{c}_1^{(3)} = \overline{c}_1^{(5)} = \overline{c}_1^{(7)} = -\frac{1}{16}(1+i),$$

$$\overline{c}_1^{(2)} = \overline{c}_1^{(4)} = \overline{c}_1^{(6)} = \overline{c}_1^{(8)} = -\frac{1}{16}(1-i),$$

$$\overline{c}_2^{(1)} = \frac{(3i+1)}{64}\int_0^{\pi} q(t)dt\,, \quad \overline{c}_2^{(2)} = \frac{(3+i)}{64}\int_0^{\pi} q(t)dt\,,$$

$$\overline{c}_2^{(3)} = \frac{(-3+i)}{64}\int_0^{\pi} q(t)dt\,, \quad \overline{c}_2^{(4)} = \frac{(-1+3i)}{64}\int_0^{\pi} q(t)dt\,,$$

$$\overline{c}_2^{(5)} = \frac{(-1-3i)}{64}\int_0^{\pi} q(t)dt\,, \quad \overline{c}_2^{(6)} = \frac{(-3-i)}{64}\int_0^{\pi} q(t)dt\,,$$

$$\overline{c}_2^{(7)} = \frac{(-i+3)}{64}\int_0^{\pi} q(t)dt\,, \quad \overline{c}_2^{(8)} = \frac{(1-3i)}{64}\int_0^{\pi} q(t)dt\,,$$

$$\overline{c}_3^{(1)} = \overline{c}_3^{(3)} = \overline{c}_3^{(5)} = \overline{c}_3^{(7)} = \frac{3(-1-i)}{128}(q(0)-q(\pi))$$

$$\overline{c}_3^{(2)} = \overline{c}_3^{(4)} = \overline{c}_3^{(6)} = \overline{c}_3^{(8)} = \frac{3(-1+i)}{128}(q(0)-q(\pi)) \tag{3. 18}$$

see figure 1 .

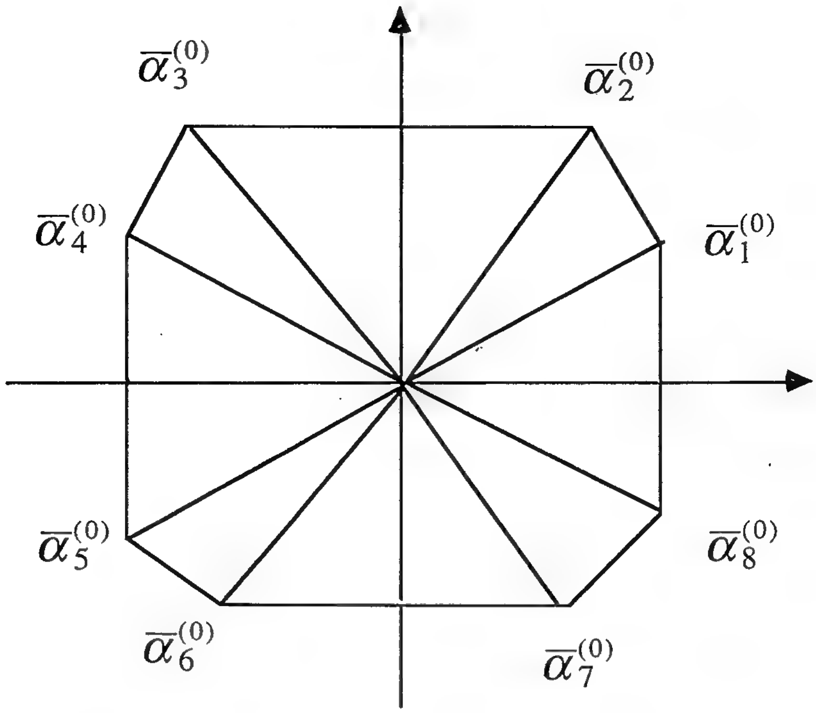



Fig.1

Using formula (3.17) , we shall obtain asymptotic formula for the eigenvalues in the following sectors

(1) If $\rho \in T_0$, then (3.17) takes the form

$$e^{\overline{\alpha}_1^{(0)}\rho\pi}\left\{\overline{c}_1^{(1)}+\frac{\overline{c}_2^{(1)}}{\rho^3}+\frac{\overline{c}_3^{(1)}}{\rho^4}+\dots\right\}+e^{\overline{\alpha}_2^{(0)}\rho\pi}\left\{\overline{c}_1^{(2)}+\frac{\overline{c}_2^{(2)}}{\rho^3}+\frac{\overline{c}_3^{(2)}}{\rho^4}+\dots\right\}=0 \qquad (3.19)$$

Formula (3.19) can be put in the following form

$$e^{(\overline{\alpha}_1^{(0)}-\overline{\alpha}_2^{(0)})\rho\pi} = -\frac{\overline{c}_1^{(2)}+\frac{\overline{c}_2^{(2)}}{\rho^3}+\frac{\overline{c}_3^{(2)}}{\rho^4}+\dots}{\overline{c}_1^{(1)}+\frac{\overline{c}_2^{(1)}}{\rho^3}+\frac{\overline{c}_3^{(1)}}{\rho^4}+\dots}$$

$$(\overline{\alpha}_1^{(0)}-\overline{\alpha}_2^{(0)})\rho\pi = (2k+\frac{1}{2})\,\pi i + \ln\left(1++\frac{\overline{c}_2^{(2)}}{\overline{c}_1^{(2)}\rho^3}++\dots\right)$$

$$-\ln\left(1++\frac{\overline{c}_2^{(1)}}{\overline{c}_1^{(1)}\rho^3}+\dots\right)$$

using formula (3.18) , we have

$$(1\text{-i})\ \rho\pi = (2k+\frac{1}{2})\ \pi i + \frac{1}{\rho^3}\left\{\frac{\overline{c}_2^{(2)}}{\overline{c}_1^{(2)}} - \frac{\overline{c}_2^{(1)}}{\overline{c}_1^{(1)}}\right\} + \frac{1}{\rho^4}\left\{\frac{\overline{c}_3^{(2)}}{\overline{c}_1^{(2)}} - \frac{\overline{c}_3^{(1)}}{\overline{c}_1^{(1)}}\right\} + \ldots$$

$$\rho_{k,0} \sim (-1+i)(k+\frac{1}{4})\left[1 - \frac{1}{16\pi(k+\frac{1}{4})^4}\int_0^\pi q(t)dt + O(\frac{1}{k^6})\right]$$

(2) If $\rho \in T_1$ , then (3.17) can be written in the form

$$e^{\overline{\alpha}_3^{(0)}\rho\pi}\left\{\overline{c}_1^{(3)} + \frac{\overline{c}_2^{(3)}}{\rho^3} + \frac{\overline{c}_3^{(3)}}{\rho^4} + \ldots\right\} + e^{\overline{\alpha}_2^{(0)}\rho\pi}\left\{\overline{c}_1^{(2)} + \frac{\overline{c}_2^{(2)}}{\rho^3} + \frac{\overline{c}_3^{(2)}}{\rho^4} + \ldots\right\} = 0 \tag{3. 20}$$

Formula (3. 20) can be written in the form

$$e^{(\overline{\alpha}_2^{(0)} - \overline{\alpha}_3^{(0)})\rho\pi} = -\frac{\overline{c}_1^{(3)} + \frac{\overline{c}_2^{(3)}}{\rho^3} + \ldots}{\overline{c}_1^{(2)} + \frac{\overline{c}_2^{(2)}}{\rho^3} + \ldots}$$

$$(\overline{\alpha}_2^{(0)} - \overline{\alpha}_3^{(0)})\ \rho\pi = (2k+\frac{1}{2})\ \pi i + \ln\left(1 + \frac{\overline{c}_2^{(3)}}{\overline{c}_1^{(3)}\rho^3} + \ldots\right) - \ln\left(1 + \frac{\overline{c}_2^{(2)}}{\overline{c}_1^{(2)}\rho^3} + \ldots\right),$$

upon using (3. 18) , we have

$$2\rho\pi = (2k+\frac{1}{2})\ \pi i + \frac{1}{\rho^3}\left\{\frac{\overline{c}_2^{(3)}}{\overline{c}_1^{(3)}} - \frac{\overline{c}_2^{(2)}}{\overline{c}_1^{(2)}}\right\} + \ldots$$

$$\rho_{k,1} \sim i(k+\frac{1}{4})\left[1 + \frac{1}{4\pi(k+\frac{1}{4})^4}\int_0^\pi q(t)dt + O(\frac{1}{k^6})\right].$$

(3) If $\rho \in T_2$ , then (3. 17) takes the form

$$\overline{\alpha}_3^{(0)\rho\pi}\left\{\overline{c}_1^{(3)}+\frac{\overline{c}_2^{(3)}}{\rho^3}+\frac{\overline{c}_3^{(3)}}{\rho^4}+\ldots\right\}+\overline{\alpha}_4^{(0)\rho\pi}\left\{\overline{c}_1^{(4)}+\frac{\overline{c}_2^{(4)}}{\rho^3}+\frac{\overline{c}_3^{(4)}}{\rho^4}+\ldots\right\}=0 \tag{3. 21}$$

Formula (3. 21) can be put in the following form

$$e^{(\overline{\alpha}_3^{(0)}-\overline{\alpha}_4^{(0)})\rho\pi} = -\frac{\overline{c}_1^{(4)}+\dfrac{\overline{c}_2^{(4)}}{\rho^3}+\ldots}{\overline{c}_1^{(3)}+\dfrac{\overline{c}_2^{(3)}}{\rho^3}+\ldots}$$

$$(\overline{\alpha}_3^{(0)}-\overline{\alpha}_4^{(0)})\rho\pi = (2\,k+\frac{1}{2})\,\pi i+\ln\left(1+\frac{\overline{c}_2^{(4)}}{\overline{c}_1^{(4)}\rho^3}+\ldots\right)$$

$$-\ln\left(1+\frac{\overline{c}_2^{(3)}}{\overline{c}_1^{(3)}\rho^3}+\ldots\right),$$

using formula (3.18.) , we get

$$(1+i)\,\rho\pi = (2\,k+\frac{1}{2})\,\pi i+\frac{1}{\rho^3}\left\{\frac{\overline{c}_2^{(4)}}{\overline{c}_1^{(4)}}-\frac{\overline{c}_2^{(3)}}{\overline{c}_1^{(3)}}\right\}+\ldots$$

$$\rho_{k,2}\sim(1+i)\,(k+\frac{1}{4})\left[1-\frac{1}{16\pi(k+\frac{1}{4})^4}\int_0^{\pi}q(t)dt+O(\frac{1}{k^6})\right].$$

(4) If $\rho \in T_3$ , then (3.17) has the following equation

$$\overline{\alpha}_4^{(0)\rho\pi}\left\{\overline{c}_1^{(4)}+\frac{\overline{c}_2^{(4)}}{\rho^3}+\ldots\right\}+\overline{\alpha}_5^{(0)\rho\pi}\left\{\overline{c}_1^{(5)}+\frac{\overline{c}_2^{(5)}}{\rho^3}+\ldots\right\}=0 \tag{3.22}$$

Formula ( 3. 22) takes the following form

$$e^{(\bar{\alpha}_4^{(0)}-\bar{\alpha}_5^{(0)})\rho\pi} = -\frac{\bar{c}_1^{(5)}+\dfrac{\bar{c}_2^{(5)}}{\rho^3}+\ldots}{\bar{c}_1^{(4)}+\dfrac{\bar{c}_2^{(4)}}{\rho^3}+\ldots}$$

$$(\bar{\alpha}_4^{(0)}-\bar{\alpha}_5^{(0)})\rho\pi = (2\mathrm{k}+\frac{1}{2})\pi\mathrm{i} + \ln(1+\frac{\bar{c}_2^{(5)}}{\bar{c}_1^{(5)}\rho^3}+\ldots)$$

$$-\ln(1+\frac{\bar{c}_2^{(4)}}{\bar{c}_1^{(4)}\rho^3}+\ldots),$$

from (3.18) , we have

$$2\pi\mathrm{i}\rho = (2\mathrm{k}+\frac{1}{2})\pi\mathrm{i} + \frac{1}{\rho^3}\left\{\frac{\bar{c}_2^{(5)}}{\bar{c}_1^{(5)}}-\frac{\bar{c}_2^{(4)}}{\bar{c}_1^{(4)}}\right\}+\ldots$$

$$\rho_{k,3} \sim (\mathrm{k}+\frac{1}{4})\left[1+\frac{1}{4\pi(k+\frac{1}{4})^4}\int_0^\pi q(t)dt+O(\frac{1}{k^6})\right].$$

(5) If $\rho\in \mathrm{T}_4$ , then (3. 17) can be written in the form

$$e^{\bar{\alpha}_6^{(0)}\rho\pi}\left\{\bar{c}_1^{(6)}+\frac{\bar{c}_2^{(6)}}{\rho^3}+\ldots\right\}+e^{\bar{\alpha}_5^{(0)}\rho\pi}\left\{\bar{c}_1^{(5)}+\frac{\bar{c}_2^{(5)}}{\rho^3}+\ldots\right\}=0\,. \quad (3.\ 23)$$

Formula ( 3. 23 ) can be written in the following form

$$e^{(\bar{\alpha}_5^{(0)}-\bar{\alpha}_6^{(0)})\rho\pi} = -\frac{\bar{c}_1^{(6)}+\dfrac{\bar{c}_2^{(6)}}{\rho^3}+\ldots}{\bar{c}_1^{(5)}+\dfrac{\bar{c}_2^{(5)}}{\rho^3}+\ldots}$$

$$(\bar{\alpha}_5^{(0)}-\bar{\alpha}_6^{(0)})\rho\pi = (2\mathrm{k}+\frac{1}{2})\pi\mathrm{i} + \ln(1+\frac{\bar{c}_2^{(6)}}{\bar{c}_1^{(6)}\rho^3}+\ldots)$$

$$-\ln(1+\frac{\bar{c}_2^{(5)}}{\bar{c}_1^{(5)}\rho^3}+\ldots),$$

using formula ( 3. 18 ) , we get

$$(-1+i)\rho\pi = (2k+\frac{1}{2})\pi i + \frac{1}{\rho^3}\left\{\frac{\overline{c}_2^{(6)}}{\overline{c}_1^{(6)}} - \frac{\overline{c}_2^{(5)}}{\overline{c}_1^{(5)}}\right\} + \ldots$$

$$\rho_{k,4} \sim (1-i)(k+\frac{1}{4})\left[1 - \frac{1}{16\pi(k+\frac{1}{4})^4}\int_0^\pi q(t)dt + O(\frac{1}{k^6})\right].$$

(6) If $\rho \in T_5$, then formula (3. 17) takes the form

$$\overline{\alpha}_7^{(0)\rho\pi}\left\{\overline{c}_1^{(7)} + \frac{\overline{c}_2^{(7)}}{\rho^3} + \ldots\right\} + \overline{\alpha}_6^{(0)\rho\pi}\left\{\overline{c}_1^{(6)} + \frac{\overline{c}_2^{(6)}}{\rho^3} + \ldots\right\} = 0. \quad (3.\ 24)$$

Formula ( 3. 24 ) can be put in the form

$$e^{(\overline{\alpha}_6^{(0)} - \overline{\alpha}_7^{(0)})\rho\pi} = -\frac{\overline{c}_1^{(7)} + \frac{\overline{c}_2^{(7)}}{\rho^3} + \ldots}{\overline{c}_1^{(6)} + \frac{\overline{c}_2^{(6)}}{\rho^3} + \ldots}$$

$$(\overline{\alpha}_6^{(0)} - \overline{\alpha}_7^{(0)})\rho\pi = (2k+\frac{1}{2})\pi i + \ln\left(1 + \frac{\overline{c}_2^{(7)}}{\overline{c}_1^{(7)}\rho^3} + \ldots\right)$$

$$- \ln\left(1 + \frac{\overline{c}_2^{(6)}}{\overline{c}_1^{(6)}\rho^3} + \ldots\right).$$

Using formula (3. 18) , we have

$$-2\pi\rho = (2k+\frac{1}{2})\pi i + \frac{1}{\rho^3}\left\{\frac{\overline{c}_2^{(7)}}{\overline{c}_1^{(7)}} - \frac{\overline{c}_2^{(6)}}{\overline{c}_1^{(6)}}\right\} + \ldots$$

$$\rho_{k,5} \sim -i(k+\frac{1}{4})\left[1 + \frac{1}{4\pi(k+\frac{1}{4})^4}\int_0^\pi q(t)dt + O(\frac{1}{k^6})\right].$$

(7) If $\rho \in T_6$, then (3. 17) can be written in the following form

$$\overline{\underset{e}{\alpha}}{}_7^{(0)}\rho\pi\left\{\bar{c}_1^{(7)}+\frac{\bar{c}_2^{(7)}}{\rho^3}+\ldots\right\}+\overline{\underset{e}{\alpha}}{}_8^{(0)}\rho\pi\left\{\bar{c}_1^{(8)}+\frac{\bar{c}_2^{(8)}}{\rho^3}+\ldots\right\}=0. \quad (3.\ 25)$$

Formula ( 3. 25 ) can be written in the following form

$$\underset{e}{(}\bar{\alpha}_7^{(0)}-\bar{\alpha}_8^{(0)})\rho\pi = -\frac{\bar{c}_1^{(8)}+\dfrac{\bar{c}_2^{(8)}}{\rho^3}+\ldots}{\bar{c}_1^{(7)}+\dfrac{\bar{c}_2^{(7)}}{\rho^3}+\ldots}$$

$$(\bar{\alpha}_7^{(0)}-\bar{\alpha}_8^{(0)})\rho\pi = (2\,k+\frac{1}{2})\,\pi i+\ln\left(1+\frac{\bar{c}_2^{(8)}}{\bar{c}_1^{(8)}\rho^3}+\ldots\right)$$

$$-\ln\left(1+\frac{\bar{c}_2^{(7)}}{\bar{c}_1^{(7)}\rho^3}+\ldots\right),$$

upon using formula (3. 18) , we have

$$(-1-i)\,\rho\pi=(2\,k+\frac{1}{2})\,\pi i+\frac{1}{\rho^3}\left\{\frac{\bar{c}_2^{(8)}}{\bar{c}_1^{(8)}}-\frac{\bar{c}_2^{(7)}}{\bar{c}_1^{(7)}}\right\}+\ldots$$

$$\rho_{k,6}\sim-(1+i)\,(k+\frac{1}{4})\left[1-\frac{1}{16\pi(k+\frac{1}{4})^4}\int_0^\pi q(t)dt+O(\frac{1}{k^6})\right].$$

(8) If $\rho\in T_7$ , then (3. 17) takes the following form

$$\overline{\underset{e}{\alpha}}{}_8^{(0)}\rho\pi\left\{\bar{c}_1^{(8)}+\frac{\bar{c}_2^{(8)}}{\rho^3}+\ldots\right\}+\overline{\underset{e}{\alpha}}{}_1^{(0)}\rho\pi\left\{\bar{c}_1^{(1)}+\frac{\bar{c}_2^{(1)}}{\rho^3}+\ldots\right\}=0. \quad (3.\ 26)$$

Formula ( 3. 26 ) takes the form

$$\underset{e}{(}\bar{\alpha}_8^{(0)}-\bar{\alpha}_1^{(0)})\rho\pi = -\frac{\bar{c}_1^{(1)}+\dfrac{\bar{c}_2^{(1)}}{\rho^3}+\ldots}{\bar{c}_1^{(8)}+\dfrac{\bar{c}_2^{(8)}}{\rho^3}+\ldots}$$

$$(\bar{\alpha}_8^{(0)}-\bar{\alpha}_1^{(0)})\rho\pi = (2\,k+\frac{1}{2})\,\pi i+\ln\left(1+\frac{\bar{c}_2^{(1)}}{\bar{c}_1^{(1)}\rho^3}+\ldots\right)$$

$$-\ln\left(1+\frac{\bar{c}_2^{(8)}}{\bar{c}_1^{(8)}\rho^3}+\ldots\right),$$

using formula (3. 18) , we have

$$-2i\rho\pi = (2k+\frac{1}{2})\pi i + \frac{1}{\rho^3}\left\{\frac{\overline{c}_2^{(1)}}{\overline{c}_1^{(1)}} - \frac{\overline{c}_2^{(8)}}{\overline{c}_1^{(8)}}\right\} + \ldots$$

$$\rho_{k,7} \sim -(k+\frac{1}{4})\left[1+\frac{1}{4\pi(k+\frac{1}{4})^4}\int_0^\pi q(t)dt + O(\frac{1}{k^6})\right] .$$

**3.3. The regularized traces of problem ( 3. 1) - (3. 2) .**

Now we define the so called " regularized trace " . The main role of this section is calculating the regularized traces of eigenvalues of problem (3. 1) - (3. 2) .

We shall calculate the regularized trace of the problem (3. 1) - (3. 2) by using the following methods :

**The first method** . Let us assume that q (x) is an infinitely differentiable function . Then from theorem (3. 1) by raising both sides to the power 4 , we have

$$\rho_{k,s}^4 = \alpha_s^4 (k+\frac{1}{4})^4\left[1+\frac{1}{\alpha_s^4(k+\frac{1}{4})^4}\cdot\frac{1}{\pi}\int_0^\pi q(t)dt + \ldots\right],$$

( s = 0 , 1 ,2 , ..., 7) . (3. 27)

From ( 3. 27) , we deduce that $\sum\limits_{k=1}^{\infty}\sum\limits_{s=0}^{7}\rho_{k,s}^4$ diverges , while

$$\sum_{k=1}^{\infty}\left[\sum_{s=0}^{7}\rho_{k,s}^4 + 12(k+\frac{1}{4})^4 - \frac{8}{\pi}\int_0^\pi q(t)dt\right], \qquad (3.\ 28)$$

converges , then we shall evaluate (3. 28) by the following theorem.

**Theorem 3. 2** : The regularized traces of problem (3. 1) - (3. 2) are given by the following formula

$$\sum_{k=1}^{\infty}\left[\sum_{s=0}^{7}\rho_{k,s}^{4}+12(k+\frac{1}{4})^{4}-\frac{8}{\pi}\int_{0}^{\pi}q(t)dt\right]=-12(q(0)-q(\pi))+\frac{4}{\pi}\int_{0}^{\pi}q(t)dt. \tag{3. 29}$$

Proof : Upon using formula ( 3. 27) , we get

$$\rho_{k,s}^{4}=\sum_{n=0}^{\infty}\frac{Q_{2n}^{(s)}}{(k+\frac{1}{4})^{2n-4}} \tag{3. 30}$$

where

$$Q_0^{(s)}=\alpha_s^4 \quad Q_2^{(s)}=0\,,\quad Q_4^{(s)}=\frac{1}{\pi}\int_0^{\pi}q(t)dt,...$$

We define the function $\psi_2$ ( - 4 ) as follows

$$\psi_2(-4)=\sum_{k=1}^{\infty}\sum_{s=0}^{7}\left[\rho_{k,s}^{4}-\sum_{l=0}^{2}\frac{Q_{2l}^{(s)}}{(k+\frac{1}{4})^{2l-4}}\right]=Z(-4)-\phi_2(-4) \tag{3. 31}$$

where , Z ( - 4 ) = $w_5$ , and

$$\phi_2(-4)=\sum_{l=0}^{2}Q_{2l}\sum_{k=l}^{\infty}\frac{1}{(k+\frac{1}{4})^{2l-4}}=\sum_{l=0}^{2}Q_{2l}.\zeta\,(2l-4,\frac{1}{4})\,,$$

$Q_{2l}=\sum_{s=0}^{7}Q_{2l}^{(s)}$, then

$$\phi_2(-4)=Q_0\zeta\,(-4,\frac{1}{4})+Q_2\zeta(-2,\frac{1}{4})+Q_4\zeta\,(0,\frac{1}{4})=-\frac{1}{2}\,Q_4\,,$$

but $\quad Q_4=\sum_{s=0}^{7}Q_4^{(s)}=\frac{8}{\pi}\int_0^{\pi}q(t)dt$,

then $\quad \phi_2(-4)=-\frac{4}{\pi}\int_0^{\pi}q(t)dt$,

Since Z ( - 4 ) = $w_5$ = -12 (q(o) - q($\pi$)), then formula ( 3. 31) takes the form

$$\psi_2(-4)=-12\,(q(0)-q(\pi))+\frac{4}{\pi}\int_0^{\pi}q(t)dt,$$

Then

$$\sum_{k=1}^{\infty}\Big[\sum_{s=0}^{7}\rho_{k,s}^{4}+12\left(k+\frac{1}{4}\right)^{4}-\frac{8}{\pi}\int_{0}^{\pi}q(t)dt\Big]$$

$$=-12\left(q(o)-q(\pi)\right)+\frac{4}{\pi}\int_{0}^{\pi}q(t)dt$$

Since $\lambda_{k,s}=\rho_{k,s}^{4}$, then the last formula can be written in the form

$$\sum_{k=1}^{\infty}\Big[\sum_{s=0}^{7}\lambda_{k,s}+12\left(k+\frac{1}{4}\right)^{4}-\frac{8}{\pi}\int_{0}^{\pi}q(t)dt\Big]$$

$$=-12\left(q(o)-q(\pi)\right)+\frac{4}{\pi}\int_{0}^{\pi}q(t)dt\,. \qquad (3.\ 32)$$

**The second method .**

Now we wish to prove formula (3. 32) by using contour integration.

First , we choose the contours $C_N^{(s)}$ ( s = 0 ,1 ,2 , ...,7) , see figure 2 .

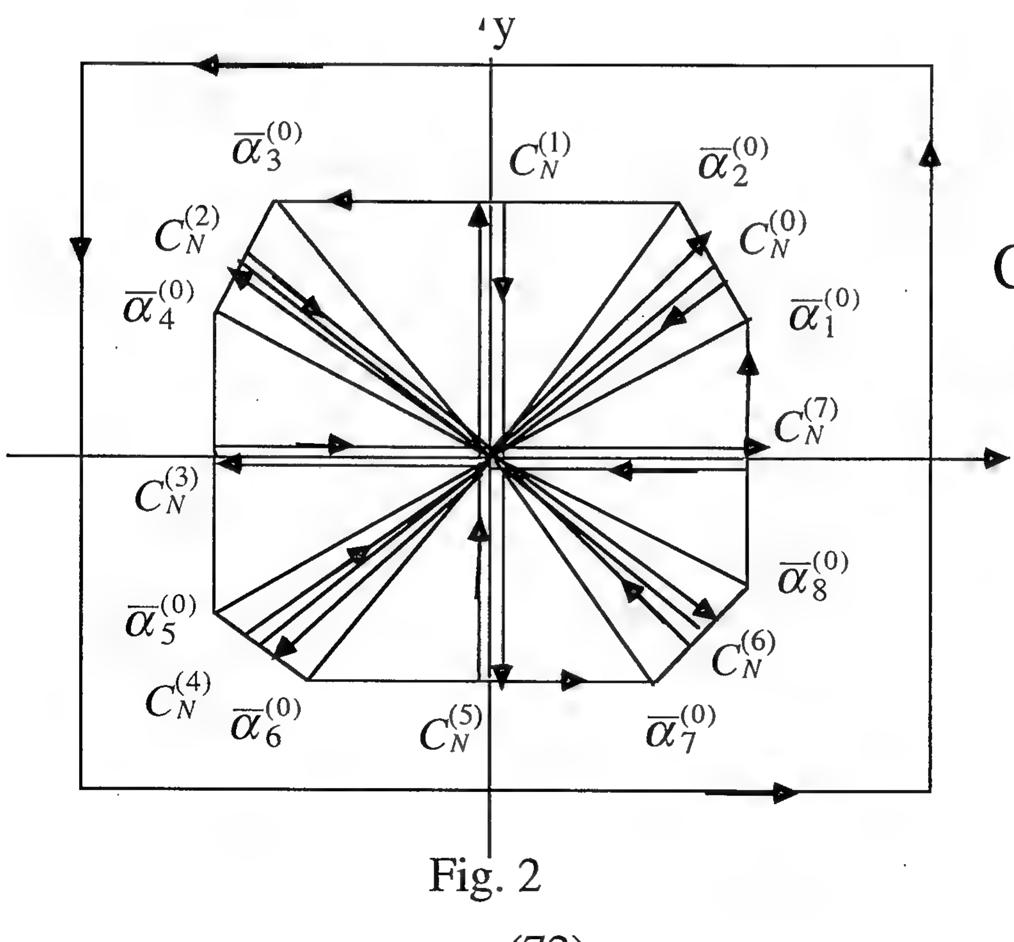



Fig. 2

From formulae (3. 16 ) , ( 3. 17) , ( 3.18) , we have

(1) If $\rho \in T_0$ , then

$$\Delta(\rho) = \left( \bar{c}_1^{(1)} e^{\bar{\alpha}_1^{(0)}\rho\pi} + \bar{c}_1^{(2)} e^{\bar{\alpha}_2^{(0)}\rho\pi} \right) + \frac{1}{\rho^3} \left( \bar{c}_2^{(1)} e^{\bar{\alpha}_1^{(0)}\rho\pi} + \bar{c}_2^{(2)} e^{\bar{\alpha}_2^{(0)}\rho\pi} \right) + \ldots$$

$$\Delta(\rho) = f_0^{(0)}(\rho) + \frac{f_1^{(0)}(\rho)}{\rho^3} + \frac{f_2^{(0)}(\rho)}{\rho^4} + \ldots$$

$$\Delta(\rho) = f_0^{(0)}(\rho) \left[ 1 + \frac{f_1^{(0)}(\rho)}{\rho^3 f_0^{(0)}(\rho)} + \frac{f_2^{(0)}(\rho)}{\rho^4 f_0^{(0)}(\rho)} + \ldots \right], \qquad (3.\ 33)$$

$$= f_0^{(0)}(\rho) \left[ 1 + \Delta_1(\rho) \right],$$

where

$$f_0^{(0)}(\rho) = \bar{c}_1^{(1)} e^{\bar{\alpha}_1^{(0)}\rho\pi} + \bar{c}_1^{(2)} e^{\bar{\alpha}_2^{(0)}\rho\pi}, \quad \Delta_1(\rho) = \frac{f_1^{(0)}(\rho)}{\rho^3 f_0^{(0)}(\rho)} + \ldots$$

The zeros of the function $f_0^{(0)}(\rho)$ are

$$\rho_{k,0} = (-1 + i)(k + \frac{1}{4}). \qquad (3.\ 34)$$

Then

$$\frac{1}{2\pi i} \oint_{C_N^{(0)}} \rho^4 \, d \ln \Delta(\rho) = \frac{1}{2\pi i} \oint_{C_N^{(0)}} \rho^4 \, d \ln f_0^{(0)}(\rho) + \frac{1}{2\pi i} \oint_{C_N^{(0)}} \rho^4 \, d \ln [\, 1 + \Delta_1(\rho) ] .$$

Since

$$\frac{1}{2\pi i} \oint_{C_N^{(0)}} \rho^4 \, d \ln \Delta(\rho) = \frac{1}{2\pi i} \oint_{C_N^{(0)}} \rho^4 \, d \ln f_0^{(0)}(\rho) - \frac{2}{\pi i} \oint_{C_N^{(0)}} \rho^3 [\, \Delta_1(\rho) - \frac{\Delta_1^2(\rho)}{2} + \ldots ] d\rho, \qquad (3.\ 35)$$

and

$$\frac{1}{2\pi i} \oint_{C_N^{(0)}} \rho^4 \, d \ln \Delta(\rho) = \sum_{k=0}^{N} \lambda_{k,0} \qquad (3.\ 36)$$

$$\frac{1}{2\pi i} \oint_{C_N^{(0)}} \rho^4 \, d \ln f_0^{(0)}(\rho) = \sum_{k=0}^{N} \rho_{k,0}^4 = -4 \sum_{k=0}^{N} (k + \frac{1}{4})^4 \qquad (3.\ 37)$$

and

$$-\frac{2}{\pi i}\oint_{C_N^{(0)}}\rho^3\,\Delta_1(\rho)d\rho = -4\Big[\sum_{k=0}^{N}\frac{f_1^{(0)}(\rho_{k,0})}{f_0'^{(0)}(\rho_{k,0})}+\frac{f_2^{(0)}(0)}{f_0^{(0)}(0)}$$

$$+\sum_{k=0}^{N}\frac{f_2^{(0)}(\rho_{k,0})}{\rho_{k,0}f_0'^{(0)}(\rho_{k,0})}+\ldots\Big]$$

$$=-4\Big[\frac{-1}{4\pi}\sum_{k=0}^{N}\int_0^{\pi}q(t)dt+\frac{3}{8}(q(0)-q(\pi))+\ldots\Big]$$

$$=\frac{1}{\pi}\int_0^{\pi}q(t)dt\sum_{k=0}^{N}1-\frac{3}{2}(q(0)-q(\pi))+\ldots \tag{3. 38}$$

by substituting (3. 36), (3. 37), (3. 38) into (3. 35), we have

$$\sum_{k=0}^{N}\rho_{k,0}^4=-4\sum_{k=0}^{N}(k+\frac{1}{4})^4+\frac{1}{\pi}\int_0^{\pi}q(t)dt\sum_{k=0}^{N}1$$

$$-\frac{3}{2}(q(0)-q(\pi))+f_0(N) \tag{3. 39}$$

(2) If $\rho\in T_1$, then

$$\Delta(\rho)=(\bar{c}_1^{(2)}e^{\bar{\alpha}_2^{(0)}\rho\pi}+\bar{c}_1^{(3)}e^{\bar{\alpha}_3^{(0)}\rho\pi})$$

$$+\frac{1}{\rho^3}(\bar{c}_2^{(2)}e^{\bar{\alpha}_2^{(0)}\rho\pi}+\bar{c}_2^{(3)}e^{\bar{\alpha}_3^{(0)}\rho\pi})+\ldots$$

$$\Delta(\rho)=f_0^{(1)}(\rho)\left[1+\frac{f_1^{(1)}(\rho)}{\rho^3f_0^{(1)}(\rho)}+\frac{f_2^{(1)}(\rho)}{\rho^4f_0^{(1)}(\rho)}+\ldots\right] \tag{3. 40}$$

The zeros of the function $f_0^{(1)}(\rho)$ are $\rho_{k,1}=i\,(k+\frac{1}{4})$ (3. 41)

$$\frac{1}{2\pi i}\oint_{C_N^{(1)}}\rho^4 d\ln\Delta(\rho)=\frac{1}{2\pi i}\oint_{C_N^{(1)}}\rho^4 d\ln f_0^{(1)}(\rho)$$

$$-\frac{4}{2\pi i}\oint_{C_N^{(1)}}\rho^3\Big[\Delta_1(\rho)-\frac{\Delta_1^2(\rho)}{2}+\ldots\Big]d\rho \tag{3. 42}$$

Since

$$\frac{1}{2\pi i}\oint_{C_N^{(1)}}\rho^4 d\ln\Delta(\rho)=\sum_{k=0}^{N}\rho_{k,1}^4, \tag{3. 43}$$

$$\frac{1}{2\pi i}\oint_{C_N^{(1)}}\rho^4 d\ln f_0^{(1)}(\rho)=\sum_{k=0}^{N}(k+\frac{1}{4})^4. \tag{3. 44}$$

And

$$-\frac{2}{\pi i}\oint_{C_N^{(1)}}\rho^4\Delta_1(\rho)d\rho=-4[\sum_{k=0}^{N}(\frac{f_1^{(1)}(\rho_{k,1})}{f_0'^{(1)}(\rho_{k,1})}+\frac{f_2^{(1)}(\rho_{k,1})}{\rho_{k,1}f_0'^{(1)}(\rho_{k,1})})+\frac{f_2^{(1)}(0)}{f_0^{(1)}(0)}\ldots]$$

$$=-4[-\frac{1}{4\pi}\sum_{k=0}^{N}\int_0^{\pi}q(t)dt+\frac{3}{8}(q(0)-q(\pi))$$

$$+\frac{3}{16\pi}(q(0)-q(\pi))\sum_{k=0}^{N}\frac{1}{(k+\frac{1}{4})}+\ldots],\quad(3.45)$$

then by substituting(3.43),(3.44) and (3.45) into(3.42),we get

$$\sum_{k=0}^{N}\rho_{k,1}^4=\sum_{k=0}^{N}(k+\frac{1}{4})^4+\frac{1}{\pi}\sum_{k=0}^{N}\int_0^{\pi}q(t)dt-\frac{3}{2}(q(0)-q(\pi))$$

$$+\frac{3}{16\pi}(q(0)-q(\pi))\sum_{k=0}^{N}\frac{1}{k+\frac{1}{4}}+f_1(N)\qquad(3.46)$$

(3) If $\rho\in T_2$, then

$$\Delta(\rho)=(\bar{c}_1^{(3)}e^{\bar{\alpha}_3^{(0)}\rho\pi}+\bar{c}_1^{(4)}e^{\bar{\alpha}_4^{(0)}\rho\pi})+\frac{1}{\rho^3}(\bar{c}_2^{(3)}e^{\bar{\alpha}_3^{(0)}\rho\pi}+\bar{c}_2^{(4)}e^{\bar{\alpha}_4^{(0)}\rho\pi})+\ldots$$

We can rewrite $\Delta(\rho)$ in the following form :

$$\Delta(\rho)=f_0^{(2)}(\rho)\left[1+\frac{f_1^{(2)}(\rho)}{\rho^3f_0^{(2)}(\rho)}+\frac{f_2^{(2)}(\rho)}{\rho^4f_0^{(2)}(\rho)}+\ldots\right]\qquad(3.47)$$

Then

$$\frac{1}{2\pi i}\oint_{C_N^{(2)}}\rho^4 d\ln\Delta(\rho)=\frac{1}{2\pi i}\oint_{C_N^{(2)}}\rho^4 d\ln f_0^{(2)}(\rho)$$

$$-\frac{2}{\pi i}\oint_{C_N^{(2)}}\rho^3[\Delta_1(\rho)-\frac{\Delta_1^2(\rho)}{2}+\ldots]d\rho,\quad(3.48)$$

but the zeros of the function $f_0^{(2)}(\rho)$ are

$$\rho_{k,2}=(1+i)(k+\frac{1}{4})\qquad(3.49)$$

Since

$$\frac{1}{2\pi i}\oint_{C_N^{(2)}}\rho^4 d\ln\Delta(\rho)=\sum_{k=0}^{N}\rho_{k,2}^4,\qquad(3.50)$$

$$\frac{1}{2\pi i}\oint_{C_N^{(2)}}\rho^4 d\ln f_0^{(2)}(\rho)=-4\sum_{k=0}^{N}(k+\frac{1}{4})^4.\qquad(3.51)$$

And

$$-\frac{4}{2\pi i}\oint_{C_N^{(2)}}\rho^3\Delta_1(\rho)d\rho=-4[\sum_{k=0}^{N}(\frac{f_1^{(3)}(\rho_{k,2})}{f_0'^{(3)}(\rho_{k,2})}+\frac{f_2^{(3)}(\rho_{k,2})}{\rho_{k,2}f_0'^{(3)}(\rho_{k,2})})+\frac{f_2^{(3)}(0)}{f_0^{(3)}(0)}...]$$

$$=\frac{1}{\pi}\sum_{k=0}^{N}\int_0^{\pi}q(t)dt-\frac{3}{2}(q(0)-q(\pi))+f_2(N)\quad(3.52)$$

Then by substituting (3. 50),(3. 51),(3. 52) into(3. 48),we have

$$\sum_{k=0}^{N}\rho_{k,2}^4=-4\sum_{k=0}^{N}(k+\frac{1}{4})^4+\frac{1}{\pi}\sum_{k=0}^{N}\int_0^{\pi}q(t)dt-\frac{3}{2}(q(0)-q(\pi))+f_2(N)$$

(3. 53)

(4) If $\rho\in T_3$ , then

$$\Delta(\rho)=(\bar{c}_1^{(4)}e^{\bar{\alpha}_4^{(0)}\rho\pi}+\bar{c}_1^{(5)}e^{\bar{\alpha}_5^{(0)}\rho\pi})+\frac{1}{\rho^3}(\bar{c}_2^{(4)}e^{\bar{\alpha}_4^{(0)}\rho\pi}+\bar{c}_2^{(5)}e^{\bar{\alpha}_5^{(0)}\rho\pi})+...$$

The function $\Delta(\rho)$ can be put in the following form

$$\Delta(\rho)=f_0^{(3)}(\rho)\left[1+\frac{f_1^{(3)}(\rho)}{\rho^3 f_0^{(3)}(\rho)}+\frac{f_2^{(3)}(\rho)}{\rho^4 f_0^{(3)}(\rho)}+...\right].\quad(3.54)$$

Then

$$\frac{1}{2\pi i}\oint_{C_N^{(3)}}\rho^4 d\ln\Delta(\rho)=\frac{1}{2\pi i}\oint_{C_N^{(3)}}\rho^4 d\ln f_0^{(3)}(\rho)$$

$$-\frac{2}{\pi i}\oint_{C_N^{(3)}}\rho^3[\Delta_1(\rho)-\frac{\Delta_1^2(\rho)}{2}+...]d\rho,\quad(3.55)$$

the zeros of the function $f_0^{(3)}(\rho)$ are $\rho_{k,3}=(k+\frac{1}{4})$ (3. 56)

Since

$$\frac{1}{2\pi i}\oint_{C_N^{(3)}}\rho^4 d\ln\Delta(\rho)=\sum_{k=0}^{N}\rho_{k,3}^4,\quad(3.57)$$

$$\frac{1}{2\pi i}\oint_{C_N^{(3)}}\rho^4 d\ln f_0^{(3)}(\rho)=\sum_{k=0}^{N}(k+\frac{1}{4})^4,\quad(3.58)$$

and

$$-\frac{2}{\pi i}\oint_{C_N^{(2)}}\rho^3\Delta_1(\rho)d\rho=-4[\sum_{k=0}^{N}(\frac{f_1^{(2)}(\rho_{k,3})}{f_0'^{(2)}(\rho_{k,3})}+\frac{f_2^{(2)}(\rho_{k,3})}{\rho_{k,3}f_0'^{(2)}(\rho_{k,3})})+\frac{f_2^{(2)}(0)}{f_0^{(2)}(0)}+...]$$

$$=\frac{1}{\pi}\sum_{k=0}^{N}\int_0^{\pi} q(t)dt-\frac{3}{2}(q(0)-q(\pi))-\frac{3}{16\pi}(q(0)-q(\pi))\sum_{k=0}^{N}\frac{1}{(k+\frac{1}{4})}+\ldots$$

(3. 59)

The substituting (3. 57),( 3. 58), ( 3. 59) into (3. 55),we have

$$\sum_{k=0}^{N}\rho_{k,3}^{4}=\sum_{k=0}^{N}(k+\frac{1}{4})^4+\frac{1}{\pi}\sum_{k=0}^{N}\int_0^{\pi} q(t)dt-\frac{3}{2}(q(0)-q(\pi))$$
$$-\frac{3}{16\pi}(q(0)-q(\pi))\sum_{k=0}^{N}\frac{1}{(k+\frac{1}{4})}+f_3(N) \qquad (3.60)$$

(5) If $\rho\in T_4$, then

$$\Delta(\rho)=(\overline{c}_1^{(5)}e^{\overline{\alpha}_5^{(0)}\rho\pi}+\overline{c}_1^{(6)}e^{\overline{\alpha}_6^{(0)}\rho\pi})+\frac{1}{\rho^3}(\overline{c}_2^{(5)}e^{\overline{\alpha}_5^{(0)}\rho\pi}+\overline{c}_2^{(6)}e^{\overline{\alpha}_6^{(0)}\rho\pi})+\ldots$$

The function $\Delta(\rho)$ can be put in the following form

$$\Delta(\rho)=f_0^{(4)}(\rho)\left[1+\frac{f_1^{(4)}(\rho)}{\rho^3 f_0^{(4)}(\rho)}+\frac{f_2^{(4)}(\rho)}{\rho^4 f_0^{(4)}(\rho)}+\ldots\right], \qquad (3.61)$$

the zeros of the function $f_0^{(4)}(\rho)$ are

$$\rho_{k,4}=-(-1+i)(k+\frac{1}{4}) \qquad (3.62)$$

Then

$$\frac{1}{2\pi i}\oint_{C_N^{(4)}}\rho^4 d\ln\Delta(\rho)=\frac{1}{2\pi i}\oint_{C_N^{(4)}}\rho^4 d\ln f_0^{(4)}(\rho)$$
$$-\frac{2}{\pi i}\oint_{C_N^{(4)}}\rho^3[\Delta_1(\rho)-\frac{\Delta_1^2(\rho)}{2}+\ldots]d\rho \qquad (3.63)$$

Since

$$\frac{1}{2\pi i}\oint_{C_N^{(4)}}\rho^4 d\ln\Delta(\rho)=\sum_{k=0}^{N}\rho_{k,4}^{4}, \qquad (3.64)$$

$$\frac{1}{2\pi i}\oint_{C_N^{(4)}}\rho^4 d\ln f_0^{(4)}(\rho)=-4\sum_{k=0}^{N}(k+\frac{1}{4})^4 \qquad (3.65)$$

and

$$-\frac{2}{\pi i}\oint_{C_N^{(4)}}\rho^3\Delta_1(\rho)d\rho=\frac{1}{\pi}\sum_{k=0}^{N}\int_0^{\pi}q(t)dt-\frac{3}{2}(q(0)-q(\pi))+\dots \quad (3.66)$$

Then substituting (3. 64), (3. 65), (3. 66) into(3. 63),we have

$$\sum_{k=0}^{N}\rho_{k,4}^4=-4\sum_{k=0}^{N}(k+\frac{1}{4})^4+\frac{1}{\pi}\sum_{k=0}^{N}\int_0^{\pi}q(t)dt$$
$$-\frac{3}{2}(q(0)-q(\pi))+f_4(N) \quad (3.67)$$

(6) If $\rho\in T_5$, then

$$\Delta(\rho)=(\bar{c}_1^{(6)}e^{\bar{\alpha}_6^{(0)}\rho\pi}+\bar{c}_1^{(7)}e^{\bar{\alpha}_7^{(0)}\rho\pi})+\frac{1}{\rho^3}(\bar{c}_2^{(6)}e^{\bar{\alpha}_6^{(0)}\rho\pi}+\bar{c}_2^{(7)}e^{\bar{\alpha}_7^{(0)}\rho\pi})+\dots$$

The function $\Delta(\rho)$ can be put in the following form

$$\Delta(\rho)=f_0^{(5)}(\rho)\left[1+\frac{f_1^{(5)}(\rho)}{\rho^3 f_0^{(5)}(\rho)}+\frac{f_2^{(5)}(\rho)}{\rho^4 f_0^{(5)}(\rho)}+\dots\right], \quad (3.68)$$

the zeros of the function $f_0^{(5)}(\rho)$ are

$$\rho_{k,5}=-i(k+\frac{1}{4}) \quad (3.69)$$

Then

$$\frac{1}{2\pi i}\oint_{C_N^{(5)}}\rho^4\, d\ln\Delta(\rho)=\frac{1}{2\pi i}\oint_{C_N^{(5)}}\rho^4\, d\ln f_0^{(5)}$$
$$-\frac{4}{2\pi i}\oint_{C_N^{(5)}}\rho^3[\Delta_1(\rho)-\dots]d\rho \quad (3.70)$$

Since

$$\frac{1}{2\pi i}\oint_{C_N^{(5)}}\rho^4\, d\ln\Delta(\rho)=\sum_{k=0}^{N}\rho_{k,5}^4, \quad (3.71)$$

$$\frac{1}{2\pi i}\oint_{C_N^{(5)}}\rho^4\, d\ln f_0^{(5)}(\rho)=\sum_{k=0}^{N}(k+\frac{1}{4})^4 \quad (3.72)$$

and

$$-\frac{4}{2\pi i}\oint_{C_N^{(5)}}\rho^3\Delta_1(\rho)d\rho=\frac{1}{\pi}\sum_{k=0}^{N}\int_0^{\pi}q(t)dt-\frac{3}{2}(q(0)-q(\pi))$$
$$-\frac{3}{16\pi}(q(0)-q(\pi))\sum_{k=0}^{N}\frac{1}{(k+\frac{1}{4})}+\dots(3.73)$$

Then substituting (3. 71), ( 3. 72),(3. 73) into(3. 70),we have

$$\sum_{k=0}^{N} \rho_{k,5}^{4} = \sum_{k=0}^{N} (k+\frac{1}{4})^4 + \frac{1}{\pi} \sum_{k=0}^{N} \int_0^{\pi} q(t)dt - \frac{3}{2}(q(0)-q(\pi))$$

$$-\frac{3}{16\pi}(q(0)-q(\pi))\sum_{k=0}^{N}\left(\frac{1}{(k+\frac{1}{4})}\right)+f_5(N) \qquad (3.74)$$

(7) If $\rho \in T_6$, then

$$\Delta(\rho)=(\overline{c}_1^{(7)} e^{\overline{\alpha}_7^{(0)}\rho\pi}+\overline{c}_1^{(8)} e^{\overline{\alpha}_8^{(0)}\rho\pi})+\frac{1}{\rho^3}(\overline{c}_2^{(7)} e^{\overline{\alpha}_7^{(0)}\rho\pi}+\overline{c}_2^{(8)} e^{\overline{\alpha}_8^{(0)}\rho\pi})+\ldots$$

The function $\Delta(\rho)$ can be put in the following form :

$$\Delta(\rho) = f_0^{(6)}(\rho)\left[1+\frac{f_1^{(6)}(\rho)}{\rho^3 f_0^{(6)}(\rho)}+\frac{f_2^{(6)}(\rho)}{\rho^4 f_0^{(6)}(\rho)}+\ldots\right], \qquad (3.75)$$

the zeros of the function $f_0^{(6)}(\rho)$ are

$$\rho_{k,6} = -(1+i)(k+\frac{1}{4}) \qquad (3.76)$$

Then

$$\frac{1}{2\pi i}\oint_{C_N^{(6)}}\rho^4\, d\ln\,\Delta(\rho) = \frac{1}{2\pi i}\oint_{C_N^{(6)}}\rho^4\, d\ln f_0^{(6)}$$

$$-\frac{4}{2\pi i}\oint_{C_N^{(6)}}\rho^3[\,\Delta_1(\rho)-\ldots\,]\,d\rho \qquad (3.77)$$

Since

$$\frac{1}{2\pi i}\oint_{C_N^{(6)}}\rho^4\, d\ln\,\Delta(\rho) = \sum_{k=0}^{N}\rho_{k,6}^4, \qquad (3.78)$$

$$\frac{1}{2\pi i}\oint_{C_N^{(6)}}\rho^4\, d\ln f_0^{(6)}(\rho) = -4\sum_{k=0}^{N}(k+\frac{1}{4})^4 \qquad (3.79)$$

and

$$-\frac{4}{2\pi i}\oint_{C_N^{(6)}}\rho^3\Delta_1\rho\, d\rho=\frac{1}{\pi}\sum_{k=0}^{N}\int_0^{\pi}q(t)dt-\frac{3}{2}(q(0)-q(\pi))+\ldots \qquad (3.80)$$

Then substituting (3. 78.),(3. 79), (3. 80) into (3.77),we get

$$\sum_{k=0}^{N}\rho_{k,6}^4=-4\sum_{k=0}^{N}(k+\frac{1}{4})^4+\frac{1}{\pi}\sum_{k=0}^{N}\int_0^{\pi}q(t)dt-\frac{3}{2}(q(0)-q(\pi))+f_6(N) \qquad (3.81)$$

(8) If $\rho \in T_7$, then

$$\Delta(\rho)=(\bar{c}_1^{(8)}\, e^{\bar{\alpha}_8^{(0)}\rho\pi}+\bar{c}_1^{(1)}\, e^{\bar{\alpha}_1^{(0)}\rho\pi})+\frac{1}{\rho^3}(\bar{c}_2^{(8)}\, e^{\bar{\alpha}_8^{(0)}\rho\pi}+\bar{c}_2^{(1)}\, e^{\bar{\alpha}_1^{(0)}\rho\pi})+\ldots$$

The function $\Delta(\rho)$ can be put in the following form :

$$\Delta(\rho) = f_0^{(7)}(\rho)\left[1+\frac{f_1^{(7)}(\rho)}{\rho^3 f_0^{(7)}(\rho)}+\frac{f_2^{(7)}(\rho)}{\rho^4 f_0^{(7)}(\rho)}+\ldots\right], \tag{3. 82}$$

the zeros of the function $f_0^{(7)}(\rho)$ are

$$\rho_{k,7} = -\,(k+\frac{1}{4}) \tag{3. 83}$$

Then

$$\frac{1}{2\pi i}\oint_{C_N^{(7)}}\rho^4\, d\ln\ \Delta(\rho)=\frac{1}{2\pi i}\oint_{C_N^{(7)}}\rho^4\, d\ln\ f_0^{(7)}(\rho)$$
$$-\frac{4}{2\pi i}\oint_{C_N^{(7)}}\rho^3\left[\ \Delta_1(\rho)-\ldots\right] d\rho \tag{3. 84}$$

Since

$$\frac{1}{2\pi i}\oint_{C_N^{(7)}}\rho^4\, d\ln\ \Delta(\rho)=\sum_{k=0}^{N}\rho_{k,7}^4, \tag{3. 85}$$

$$\frac{1}{2\pi i}\oint_{C_N^{(7)}}\rho^4\, d\ln\ f_0^{(7)}(\rho)=\sum_{k=0}^{N}(k+\frac{1}{4})^4 \tag{3. 86}$$

and

$$-\frac{2}{\pi i}\oint_{C_N^{(7)}}\rho^3\,\Delta_1(\rho)\, d\rho=\frac{1}{\pi}\sum_{k=0}^{N}\int_0^{\pi}q(t)dt-\frac{3}{2}(q(0)-q(\pi))$$
$$+\frac{3}{16\pi}(q(0)-q(\pi))\sum_{k=0}^{N}\frac{1}{(k+\frac{1}{4})}+f_{7(N)}\,. \tag{3.87}$$

Then substituting (3. 85), ( 3. 86), (3. 87) into (3. 84) ,we get

$$\sum_{k=0}^{N}\rho_{k,7}^4=\sum_{k=0}^{N}(k+\frac{1}{4})^4+\frac{1}{\pi}\sum_{k=0}^{N}\int_0^{\pi}q(t)dt-\frac{3}{2}(q(0)-q(\pi))$$
$$+\frac{3}{16\pi}(q(0)-q(\pi))\sum_{k=0}^{N}\frac{1}{k+\frac{1}{4}}+f_{7(N)}\,. \tag{3. 88}$$

From formulae (3. 39) , (3. 46) , (3. 53.) , (3. 60) , (3. 67), (3. 74), (3. 81) and (3. 88), we have

$$\sum_{k=0}^{N}\sum_{s=0}^{7}\lambda_{k,s}=-12\sum_{k=0}^{N}(k+\frac{1}{4})^4+\frac{8}{\pi}\sum_{k=0}^{N}\int_0^{\pi}q(t)dt-12(q(0)-q(\pi))+f(N), \tag{3. 89}$$

where

$$f(N)=\sum_{s=0}^{7}f_s(N),$$

$$f_0(N)=\frac{1}{\pi i}\oint_{C_N^{(0)}}\frac{f_1^{2(0)}(\rho)}{\rho^3 f_0^{2(0)}(\rho)}d\rho+\ldots$$

Now we shall prove that $\lim_{N\to\infty} f_0(N)=0$

Since

$$\frac{1}{\pi i}\oint_{C_N^{(0)}}\frac{f_1^{2(0)}(\rho)}{\rho^3 f_0^{2(0)}(\rho)}d\rho=2\{\frac{1}{2}[2\frac{f_1^{(0)}(0)f_1''^{(0)}(0)}{f_{(0)}^{2(0)}(0)}$$

$$+2\frac{f_1'^{2(0)}(0)}{f_0^{2(0)}(0)}-8\frac{f_1^{(0)}(0)f_1'^{(0)}(0)f_0'^{(0)}(0)}{f_{(0)}^{3(0)}(0)}-2\frac{f_0''^{(0)}(0)f_1^{2(0)}(0)}{f_{(0)}^{3(0)}(0)}+6\frac{f_1^{2(0)}(0)f_0'^{2(0)}(0)}{f_{(0)}^{4(0)}(0)}$$

$$+\sum_{k=0}^{N}[-2\frac{f_1^{2(0)}(\rho_{k,0})\psi'_1(\rho_{k,0})}{\rho_{k,0}^3\psi_1^3(\rho_{k,0})}+2\frac{f_1^{(0)}(\rho_{k,0})f_1'^{(0)}(\rho_{k,0})}{\rho_{k,0}^3\psi_1^2(\rho_{k,0})}$$

$$-3\frac{f_1^{2(0)}(\rho_{k,0})}{\rho_{k,0}^4\psi_1^2(\rho_{k,0})}]\} \tag{3. 90}$$

Where

$$f_0^{(0)}(\rho) = (\rho - \rho_{k,0})\psi_1(\rho)$$

Since

$$f_0^{(0)}(0) = -\frac{1}{8},\ f'^{(0)}_0(0) = -\frac{\pi(1+i)}{4},\ f''^{(0)}_0(0) = \frac{7\pi^2 i}{8},$$

$$f_1^{(0)}(0) = \frac{(1+i)}{16}\int_0^\pi q(t)dt,\ f'^{(0)}_1(0) = -\frac{7\pi i}{32}\int_0^\pi q(t)dt.,$$

$$f''^{(0)}_1(0) = \frac{11\pi^2}{32}(-1+\mathrm{i})\int_0^\pi q(t)dt,\ f_0^{(0)}(\rho_{k,0}) = 0,$$

$$f'^{(0)}_0(\rho_{k,0}) = -\frac{\pi(-)^k(1+i)}{8\sqrt{2}}\,\bar{e}^{3(k+\frac{1}{4})\pi},$$

$$f''^{(0)}_0(\rho_{k,0}) = -\frac{3\pi^2(-)^k i}{4\sqrt{2}}\,\bar{e}^{3(k+\frac{1}{4})\pi},$$

$$f_1^{(0)}(\rho_{k,0}) = \frac{(-)^k(1+i)\,\bar{e}^{3(k+\frac{1}{4})\pi}}{32\sqrt{2}}\int_0^\pi q(t)dt,$$

and

$$f'^{(0)}_1(\rho_{k,0}) = \frac{3\pi(-)^k i}{16\sqrt{2}}\,\bar{e}^{3(k+\frac{1}{4})\pi}\int_0^\pi q(t)dt \qquad (3.91)$$

Then substituting (3.91) into (3. 90) , we have

$$\frac{1}{\pi i}\oint_{C_N^{(\rho)}}\frac{f_1^{2(0)}(\rho)}{\rho^3 f_0^{2(0)}(\rho)}\,d\rho = 2\{\frac{\pi^2}{16}\int_0^\pi q(t)dt[-44-49+224+56-192]$$

$$+\frac{3\int_0^\pi q(t)dt}{32\pi}\sum_{k=0}^{N}\frac{1}{(k+\frac{1}{4})^3}+\frac{3}{64\pi^2}\int_0^\pi q(t)dt\sum_{k=0}^{N}\frac{1}{(k+\frac{1}{4})^4}\}$$

let $N\to\infty$, then we get

$$\lim_{N\to\infty}\frac{1}{\pi i}\oint_{C_N^{(1)}}\frac{f_1^{2(0)}(\rho)}{\rho^3 f_0^{2(0)}(\rho)}\,d\rho$$

$$= 2\{-\frac{5\pi^2}{16}\int_0^\pi q(t)dt + \frac{3}{32\pi}\int_0^\pi q(t)dt\,.\sum_{k=0}^{\infty}\frac{1}{(k+\frac{1}{4})^3}$$

$$+\frac{3}{64\pi^2}\int_0^{\pi} q(t)dt.\ \sum_{k=0}^{\infty}\frac{1}{(k+\frac{1}{4})^{4}}\}$$

Since

$$\sum_{k=0}^{\infty}\frac{1}{(k+\frac{1}{4})^{3}} = 2\pi^3,\ \sum_{k=0}^{\infty}\frac{1}{(k+\frac{1}{4})^{4}} = \frac{8\pi^4}{3}$$, see [29].

Then

$$\lim_{N\to\infty}\frac{1}{\pi i}\oint_{C_N^{(1)}}\frac{f_1^{2(0)}(\rho)}{\rho^3 f_0^{2(0)}(\rho)}\,d\rho = 2\{-\frac{5\pi^2}{16}\int_0^{\pi} q(t)dt + \frac{3\pi^2}{16}\int_0^{\pi} q(t)dt + \frac{2\pi^2}{16}\int_0^{\pi} q(t)dt\} = 0$$

From the last formula , we have

$$\lim_{N\to\infty} f_0(N) = 0 .$$

By that same way ,we can prove that $\lim_{N\to\infty} f_s(N) = 0$ ( $s=1,2,\dots,7$)

Then so as $N\to\infty$ , formula (3. 89) takes the form

$$\sum_{k=1}^{\infty}\left[\sum_{s=0}^{7}\lambda_{k,s} + 12\,(k+\frac{1}{4})^4 - \frac{8}{\pi}\int_0^{\pi} q(t)dt\right] = \frac{4}{\pi}\int_0^{\pi} q(t)dt - 12\,(q(0) - q(\pi)) .$$

**3.4 . The regularized sum of the eigenfunction of problem (3. 1) - (3. 2) .**

There is an interesting connection between the eigenfunction and the Green's function . Hence , in the following we obtained the Green's function of the problem (3. 1) - (3. 2) and by using it we can find the regularized sum for the eigenfunction .

**Lemma 3. 1** : The Gree$n's$ function of problem (3.1) - (3.2) is given by the following formula

$$G(x,\xi,\rho) = \frac{H(x,\xi,\rho)}{\Delta(\rho)},$$

where

$$\begin{aligned}\Delta(\rho) = {} & 4 + 4\cosh\rho\pi\cos\rho\pi - \frac{7}{2}(\cos\rho\pi + \cosh\rho\pi) \\ & - \frac{\cos\rho\pi}{2}(\sin\rho\pi\sinh\rho\pi + \cos\rho\pi\cosh\rho\pi) \\ & + \frac{\cosh\rho\pi}{2}(\sin\rho\pi\sinh\rho\pi - \cos\rho\pi\cosh\rho\pi) \\ & + \frac{\int_o^\pi q(t)dt}{16\rho^3}\Big[\,12(\sinh\rho\pi - \sin\rho\pi) \\ & + 16(\cosh\rho\pi\sin\rho\pi - \sinh\rho\pi\cos\rho\pi) \\ & - \cosh\rho\pi\sin\rho\pi\,(6\cosh\rho\pi + 2\cos\rho\pi) \\ & + \sinh\rho\pi\cos\rho\pi\,(6\cos\rho\pi + 2\cosh\rho\pi)\Big] \\ & + \frac{(q(0)-q(\pi))}{32\rho^4}\Big[6\sin\rho\pi\cosh\rho\pi(\sin\rho\pi + \sinh\rho\pi) \\ & - 6\sinh\rho\pi\cos\rho\pi(\sin\rho\pi + \sinh\rho\pi)\Big] + o\left(\frac{1}{\rho^6}\right),\end{aligned}$$

and

$H(x,\xi,\rho) =$

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & \phi_3(x) & \phi_4(x) & g(x,\xi) \\ \phi_1(\pi)-1 & \phi_2(\pi) & \phi_3(\pi) & \phi_4(\pi) & g(\pi,\xi)-g(0,\xi) \\ \phi'_1(\pi) & \phi'_2(\pi)-1 & \phi'_3(\pi) & \phi'_4(\pi) & g'(\pi,\xi)-g'(0,\xi) \\ \phi''_1(\pi) & \phi''_2(\pi) & \phi''_3(\pi)-1 & \phi''_4(\pi) & g''(\pi,\xi)-g''(0,\xi) \\ \phi'''_1(\pi) & \phi'''_2(\pi) & \phi'''_3(\pi) & \phi'''_4(\pi)-1 & g'''(\pi,\xi)-g'''(0,\xi) \end{vmatrix} \quad (3.92)$$

**Lemma 3 . 2** : For the function g (x, $\xi$) and its derivatives with respect to x has the following asymptotic formulae

$$g(x,\xi) = \pm \frac{1}{2}\{ \frac{1}{2\rho^3}\ [\sin\rho(\xi - x) + \sinh\rho(x - \xi)]$$

$$+ \frac{1}{8\rho^6} \int_0^{\xi} q(t)dt\ [\cosh\rho\,(x - \xi) - \cos\rho\,(x - \xi)]$$

$$+ \frac{1}{8\rho^6} \int_0^{x} q(t)dt\ [\cos\rho\,(x - \xi) - \cosh\rho\,(x - \xi)]$$

$$+ \frac{3}{16\rho^7}(q(x) + q(\xi))\,[\sin\rho(\xi - x) + \sinh\rho\,(x - \xi)] + \ldots\} \qquad (3.93)$$

$$g'(x,\xi) = \pm \frac{1}{2}\{ \frac{1}{2\rho^2}\ [\cosh\rho\,(x - \xi) - \cos\rho\,(x - \xi)]$$

$$+ \frac{1}{8\rho^5} \int_0^{\xi} q(t)dt\ [\sin\rho\ (x - \xi) + \sinh\rho\,(x - \xi)]$$

$$- \frac{1}{8\rho^5} \int_0^{x} q(t)dt\ [(\sin\rho\ (x - \xi) + \sinh\rho\,(x - \xi)]$$

$$+ \frac{1}{16\rho^6}(q(x) + 3q(\xi))\,[\cosh\rho\,(x - \xi) - \cos\rho\,(x - \xi)]$$

$$+ \frac{3}{16\rho^7}\, q'(x)\,[(\sin\rho(\xi - x) + \sinh\rho\,(x - \xi)] + \ldots\ \} \qquad (3.94)$$

$$g''(x,\xi) = \pm \frac{1}{2}\{ \frac{1}{2\rho}\ [(\sin\rho\,(x - \xi) + \sinh\rho\,(x - \xi)]$$

$$+ \frac{1}{8\rho^4} \int_0^{\xi} q(t)dt\ [\cos\rho\ (x - \xi) + \cosh\rho\,(x - \xi)]$$

$$- \frac{1}{8\rho^4} \int_0^{x} q(t)dt\ [\cos\rho\ (x - \xi) + \cosh\rho\,(x - \xi)]$$

$$- \frac{1}{16\rho^5}(q(x) - 3q(\xi))\,[(\sin\rho\ (x - \xi) + \sinh\rho(x - \xi)]$$

$$+\frac{1}{4\rho^6}\, q'(x)\left[\cosh\rho\,(x-\xi)-\cos\rho\,(x-\xi)\right]$$

$$+\frac{3}{16\rho^7}\, q''(x)[(\sinh\rho(x-\xi)-\sin\rho(x-\xi)]+\ldots\Big\} \qquad (3.95)$$

and

$$g'''(x,\xi)=\pm\frac{1}{2}\Big\{\frac{1}{2}\left[\cos\rho\,(x-\xi)+\cosh\rho\,(x-\xi)\right]$$

$$+\frac{1}{8\rho^3}\int_0^{\xi} q(t)dt\left[(\sinh\rho\,(x-\xi)-\sin\rho(x-\xi)\right]$$

$$-\frac{1}{8\rho^3}\int_0^{x} q(t)dt\left[(\sinh\rho\,(x-\xi)-\sin\rho(x-\xi)\right]$$

$$-\frac{3}{16\rho^4}\,(q(x)-q(\xi))\left[\cos\rho(x-\xi)+\cosh\rho\,(x-\xi)\right]$$

$$+\frac{3}{16\rho^5}\, q'(x)\left[(\sin\rho\,(x-\xi)+\sinh\rho(x-\xi)\right]$$

$$+\frac{7}{16\rho^6}\, q''(x)\left[\cosh\rho\,(x-\xi)-\cos\rho\,(x-\xi)\right]$$

$$+\frac{3}{16\rho^7}\, q'''(x)\left[\sinh\rho\,(x-\xi)-\sin\rho(x-\xi)\right]+\ldots\Big\} \qquad (3.96)$$

Suppose that $R(x,\xi,\rho_{k,s})$ is the residue of the Green's function $G(x,\xi,\rho)$ at the points $\rho=\rho_{k,s}$ ( $s = 0, 1, 2, \ldots, 7$ ). Upon using lemma ( 3. 1) we have

$$R(x,\xi,\rho_{k,s})=\lim_{\rho\to\rho_{k,s}}\frac{(\rho-\rho_{k,s})H(x,\xi,\rho)}{\Delta(\rho)}=\frac{H(x,\xi,\rho_{k,s})}{\Delta'(\rho_{k,s})} \qquad (3.97)$$

We can obtain $\Delta'(\rho_{k,s})$ by differentiate formula (3. 15) with respect to $\rho$ .

**Theorem 3. 3** : The sum of eigenfunctions of problem (3. 1) - ( 3. 2) are given by the following formula

$$\sum_{k=1}^{\infty}\sum_{s=0}^{7}\left[\rho_{k,s}^{4}\; R(x,\xi,\rho_{k,s})\right]=0 \qquad (3.98)$$

Proof :

(1) If $\rho_k \in S_0$, then

$$\Delta'(\rho_{k,0}) \sim \frac{\pi}{16}(-1+3i)\ e^{(2-i)\pi\rho_{k,0}}\left[1+\frac{(22+4i)}{40\rho_{k,0}^3}\int_0^\pi q(t)\,dt+\ldots\right]$$

Upon using formula (3. 97) , we have

$$\rho_{k,0}^4 R(x,\xi,\rho_{k,0}) \sim \rho_{k,0}^4 \left\{\frac{e^{\alpha_j\rho_{k,0}}}{\pi(-1+3i)}\left[\frac{1}{16\rho_{k,0}^3} + \frac{1}{\rho_{k,0}^6}\left(\frac{1}{64}\left(\int_0^\pi q(t)dt+\int_0^x q(t)dt-\int_0^\xi q(t)dt\right) - \frac{(22+4i)}{640}\int_0^\pi q(t)dt\right)+\left(\frac{(22+4i)}{40}\int_0^\pi q(t)dt\right)^2+\ldots\right]\right\}.$$

$$\rho_{k,0}^4 R(x,\xi,\rho_{k,0}) \sim \frac{\rho_{k,0}\,e^{\alpha_j\rho_{k,0}}}{16\pi(-1+3i)}$$

where

$$\alpha_1 = x+\xi-\pi, \quad \alpha_2 = x-\xi$$

(2) If $\rho_k \in S_1$, then

$$\Delta'(\rho_{k,1}) \sim \frac{\pi}{16}(-3+i)\ e^{(1-2i)\pi\rho_{k,1}}\left[1+\frac{(24i-2)}{40\rho_{k,1}^3}\int_0^\pi q(t)\,dt + \frac{3}{4\pi\rho_{k,1}^4}\int_0^\pi q(t)\,dt+\ldots\right]$$

Using the forrmula ( 3 . 97) , we get

$$\rho_{k,1}^4 R(x,\xi,\rho_{k,1}) \sim \rho_{k,1}^4 \left\{\frac{\bar{e}^{i\alpha_j\rho_{k,1}}}{\pi(-3+i)}\left[\frac{i}{16\rho_{k,1}^3} + \frac{1}{\rho_{k,1}^6}\left[\frac{(2i+24)}{640}\int_0^\pi q(t)\,dt+\frac{1}{64}\left(-(2+i)\int_0^\pi q(t)\,dt - \int_0^x q(t)\,dt+\int_0^\xi q(t)\,dt\right)+\left(\frac{(-2+24i)}{40}\int_0^\pi q(t)\,dt\right)^2\right]+\ldots\right\}$$

$$\rho_{k,1}^4 R(x,\xi,\rho_{k,1}) \sim \frac{i\rho_{k,1}}{16\pi(-3+i)}\ \bar{e}^{i\alpha_j\rho_{k,1}}$$

(3) If $\rho_k \in S_2$ , then

$$\Delta'(\rho_{k,2}) \sim \frac{\pi(3+i)}{16} \overline{e}^{(1+2i)\pi\rho_{k,2}} [1 + \frac{(14+22i)}{40\rho_{k,2}^3} \int_0^\pi q(t)dt + \frac{3}{4\pi\rho_{k,2}^4} \int_0^\pi q(t)\, dt \ + ... ]$$

Using the forrmula ( 3 . 97) , we get

$$\rho_{k,2}^4 \, R(x, \xi, \rho_{k,2}) \sim i\, \rho_{k,2} \frac{\overline{e}^{i\alpha j \rho_{k,2}}}{16\pi(3+i)}$$

(4) If $\rho_k \in S_3$ then

$$\Delta'(\rho_{k,3}) \sim \frac{\pi(1+3i)}{16} \overline{e}^{(2+i)\pi\rho_{k,3}} [1 - \frac{(4-10i)}{40\rho_{k,3}^3} \int_0^\pi q(t)dt + \frac{3}{4\pi\rho_{k,3}^4} \int_0^\pi q(t)\, dt \ + ... ]$$

Using the forrmula ( 3 . 97) , we get

$$\rho_{k,3}^4 \, R(x, \xi, \rho_{k,3}) \sim \frac{\rho_{k,3} \overline{e}^{\alpha j \rho_{k,3}}}{16\pi(1+3i)}$$

(5) If $\rho_k \in S_4$ , then

$$\Delta'(\rho_{k,4}) \sim \frac{\pi(1-3i)}{16} ( \overline{e}^{2+i)\pi\rho_{k,4}} [1 - \frac{(-12+14i)}{40\rho_{k,4}^3} \int_0^\pi q(t)dt + \frac{3}{4\pi\rho_{k,4}^4} \int_0^\pi q(t)\, dt \ + ... ]$$

Using the forrmula ( 3 . 97) , we get

$$\rho_{k,4}^4 \, R(x, \xi, \rho_{k,4}) \sim \frac{\rho_{k,4} \overline{e}^{\alpha j \rho_{k,4}}}{16\pi(1-3i)}$$

(6) If $\rho_k \in S_5$ , then

$$\Delta'(\rho_{k,5}) \sim \frac{\pi(3-i)}{16} e^{(-1+2i)\pi\rho_{k,5}} [1 + \frac{(10-20i)}{40\rho_{k,5}^3} \int_0^\pi q(t)dt + \frac{3}{4\pi\rho_{k,5}^4} \int_0^\pi q(t)\, dt \ + ... ]$$

Using the forrmula ( 3 . 97) , we get

$$\rho_{k,5}^{4}\,\mathrm{R}(x,\ \xi,\rho_{k,5}) \sim \frac{i\rho_{k,5}e^{i\alpha j\rho_{k,5}}}{16\pi(-3+i)}$$

(7) If $\rho_k \in S_6$ , then

$$\Delta'(\rho_{k,6}) \sim -\frac{\pi(3+i)}{16}e^{\pi(1+2i)\pi\rho_{k,6}}\Big[1-\frac{(10+20i)}{40\rho_{k,6}^{3}}\int_0^{\pi} q(t)dt + \frac{3}{4\pi\rho_{k,6}^{4}}\int_0^{\pi} q(t)\,dt + \dots\Big]$$

Using the forrmula ( 3 . 97) , we get

$$\rho_{k,6}^{4}\,\mathrm{R}(x,\ \xi,\rho_{k,6}) \sim \frac{i\rho_{k,6}e^{i\alpha j\rho_{k,6}}}{16\pi(3+i)}$$

(8) If $\rho_k \in S_7$ , then

$$\Delta'(\rho_{k,7}) \sim -\frac{\pi(1+3i)}{16}e^{(2+i)\pi\rho_{k,7}}\Big[1-\frac{(20+10i)}{40\rho_{k,7}^{3}}\int_0^{\pi} q(t)dt + \frac{3}{4\pi\rho_{k,7}^{4}}\int_0^{\pi} q(t)\,dt + \dots\Big]$$

Using the forrmula ( 3 . 97) , we get

$$\rho_{k,7}^{4}\,\mathrm{R}(x,\ \xi,\rho_{k,7}) \sim \frac{\rho_{k,7}e^{\alpha j\rho_{k,7}}}{16\pi(3i+1)}$$

We choose $\alpha_1$ if ( $x+\xi > \pi$ , $\xi > x$ ) or ( $x+\xi > \pi$ , $\xi < x$ , $2\xi > \pi$ ) , while we choose $\alpha_2$ if ( $x+\xi > \pi$ , $\xi < x$, $\pi > 2\xi$) or ( $x+\xi < \pi$ , $\xi < x$ )

But $\rho_{k,0} = -\rho_{k,4}$ , $\rho_{k,1} = -\rho_{k,5}$ , $\rho_{k,2} = -\rho_{k,6}$ , $\rho_{k,3} = -\rho_{k,7}$ so we have :

$$\sum_{k=1}^{\infty}\sum_{s=0}^{7}\big[\,\rho_{k,s}^{4}\ \mathrm{R}(x,\ \xi,\ \rho_{k,s})\big] = 0\,.$$

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